A new estimate on Evans' Weak KAM approach

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Abstract

We consider a recent approximate variational principle for weak KAM theory proposed by Evans. As in the case of classical integrability, for one dimensional mechanical Hamiltonian systems all the computations can be carried out explicitly. In this setting, we illustrate the geometric content of the theory and prove new lower bounds for the estimates related to its dynamic interpretation. These estimates also extend to the case of n degrees of freedom.

Keywords: Weak KAM theory, Hamilton-Jacobi equation, approximate variational principles, perturbation theory.

1 Introduction

The integrability of classical mechanical systems follows from the existence of regular global solutions to the Hamilton-Jacobi equation

$$H(\tilde{p} + \frac{\partial u}{\partial q}, q) = \bar{H}(\tilde{p}) \tag{1}$$

where both the generating function $u(\tilde{p}, q)$ and the Hamiltonian $\bar{H}(\tilde{p})$ are unknown. It is well known (by the Liouville-Arnol'd Theorem) that global solutions to this problem exist only for a very special class of mechanical systems, namely, those having a complete set of first integrals. Although most mechanical systems are not integrable in this sense, many are quasi-integrable, that is they have the form

$$H(I, \varphi) = h(I) + \varepsilon f(I, \varphi), \tag{2}$$

where $(I, \varphi) \in \mathbb{R}^n \times \mathbb{T}^n$ are action-angle variables. The new approach to Hamiltonian perturbation theories motivated by Poincaré contributions culminated with the celebrated KAM theorem [1], [12], [18].

Since the early 1980's alternative approaches to the study of non-integrable Hamiltonians based on variational methods and PDE techniques [15, 16, 17], [14], [8, 9] have led to the formulation of the so-called weak KAM theory. Within the Tonelli setting, that is, assuming positive definite superlinear Lagrangians and Hamiltonians, the main results of this theory are the existence of invariant (action-minimizing) sets, generalizing KAM tori, and the existence of global weak solutions to the Hamilton-Jacobi equation (1). In particular, it has been proved in various contexts (homogenization [13], variational and viscosity [10]) that if $H(I, \varphi)$ is Tonelli, then for any I the Hamilton-Jacobi problem

$$H(I + \frac{\partial u}{\partial \varphi}, \varphi) = \bar{H}(I) \tag{3}$$

admits Lipschitz continuous solutions, with "effective" Hamiltonian

$$\bar{H}(I) := \inf_{u \in C^1(\mathbb{T}^n)} \max_{\varphi \in \mathbb{T}^n} H(I + \frac{\partial u}{\partial \varphi}, \varphi). \tag{4}$$

In the terminology of [8], these solutions are called weak KAM. Most of the dynamic interpretations of these weak solutions have been related to Aubry-Mather theory (see for example [2], [11] and the references therein).

The starting point of the present paper is an innovative formulation of weak KAM theory given by Evans [5, 6]. The main outcome of this new variational construction, inspired by Aronsson's variational principle, is a sequence of smooth functions $u_k(\tilde{I}, \varphi)$ which define, for any value of the parameters $\tilde{I}$, a dynamics and a density measure $\sigma_k(\tilde{I}, \varphi)$ on the torus $\mathbb{T}^n$. The convergence of this torus dynamics to a linear flow is expressed precisely through the asymptotic formula (2.22) in [6]. Moreover, an estimate of how the torus flow approximates the genuine Hamiltonian flow of $H(I, \varphi)$ is expressed through the asymptotic formula (2.21) of [6]. We refer to Section 2 for complete details. The fundamental relations (2.21) and (2.22) of [6] are expressed in the form of upper bounds.

The first goal of this paper is to offer a detailed geometric and dynamic representation, summarized in Figures 1 and 2, of several evolutions and flows arising from Evans framework. Moreover, we complete the fundamental estimates of Evans, (2.21) and (2.22), by also measuring the gap d_k between the original Hamiltonian flow and a crucial approximate dynamics introduced by Evans.

We remark that in the generic n dimensional case, there exists no explicit expression for the $u_k(\tilde{I}, \varphi)$, although numerical approximations may be obtained via a finite difference scheme [7]. There is one case in which the sequences u_k have an explicit analytic representation and a simple mechanical interpretation, namely, the case of one degree of freedom. In particular, in this paper we show that for a mechanical system

$$H(I, \varphi) := \frac{I^2}{2} + f(\varphi), \tag{5}$$

with $(I, \varphi) \in \mathbb{R}^+ \times \mathbb{S}^1$, the functions u_k are precisely the solutions of the Hamilton-Jacobi equations for the modified Hamiltonians

$$H_k(I, \varphi) = \frac{I^2}{2} + f(\varphi) + \frac{1}{k} \log I. \tag{6}$$

As a consequence of the special form of the term $\frac{1}{k} \log I$, for any $I \in \mathbb{R}^+$ the solution u_k of the Hamilton-Jacobi equation is explicit up to quadratures of elementary functions and the special Lambert function. By taking advantage of this explicit analytic expression for the u_k, we can prove better convergence properties than the more general ones given in [6], give new lower bounds in the inequalities (2.21), (2.22) of

[6] and also exhibit an explicit example of singular convergence of the measures σ_k.
These new lower bounds constrain the σ_k-convergence of the approximate dynamics to a linear flow to be, in general, no faster than $1/k^2$. In Section 4, we see that these one dimensional estimates also have further consequences in the integrable n dimensional case, with $n-1$ ignorable variables.

We also remark that the present one dimensional study may provide a basis for a perturbation approach to single resonances in Hamiltonian systems, whose normal forms are represented by perturbations of the mechanical pendulum.

The paper is organized as follows. In Section 2 we review the fundamentals of the Evans theory and we offer a geometric and dynamic representation of several evolutions and flows of this framework. Moreover, we measure the gap between the original Hamiltonian flow and a crucial approximate dynamics introduced by Evans. Section 3 is devoted to explicit solutions and convergence results in the one dimensional case. In Section 4, by exploiting our explicit knowledge of the sequences u_k and σ_k in the one dimensional case, we first propose refined asymptotic estimates for the integrals involved in formulas (2.21), (2.22) of [6] –integrals (18) and (19) here– and then we discuss some consequences in the quadrature-integrable n dimensional case. Sections 5, 6 and 7 are devoted to the proofs. In Section 8 we review some properties of the special Lambert function.

2 Dynamic picture of Evans theory

In [5, 6] Evans introduces a new variational version of weak KAM theory, whose outcome is a sequence of functions $u_k(\tilde{I},\varphi)$ which define, for any value of the parameter $\tilde{I}\in\mathbb{R}^n$ and any index $k\in\mathbb{N}$, a dynamics and a density measure $\sigma_k(\tilde{I},\varphi)$ on the torus $\mathbb{T}^n$. The properties of this torus dynamics and its relations with the original Hamiltonian flow represent the dynamic interest of the theory.

More precisely, instead of looking for minimizers $u(\tilde{I},\varphi)$ for the sup-norm of $H(\tilde{I}+\frac{\partial u}{\partial\varphi},\varphi)$ over $\mathbb{T}^n$, Evans looks for minimizers $u_k(\tilde{I},\varphi)$ of the functional

$$I_k[u] := \int_{\mathbb{T}^n} e^{kH(\tilde{I}+\frac{\partial u}{\partial\varphi},\varphi)}d\varphi. \tag{7}$$

Under suitable hypotheses[1] on H, the minimizers u_k turn out to be smooth and uniquely defined when one requires that $\int_{\mathbb{T}^n} u_k d\varphi = 0$. After defining the density measure over $\mathbb{T}^n$

$$\sigma_k(\tilde{I},\varphi) := e^{k(H(\tilde{I}+\frac{\partial u_k}{\partial\varphi},\varphi)-\bar{H}_k(\tilde{I}))} \tag{8}$$

where

$$\bar{H}_k(\tilde{I}) := \frac{1}{k}\log\int_{\mathbb{T}^n} e^{kH(\tilde{I}+\frac{\partial u_k}{\partial\varphi},\varphi)}d\varphi, \tag{9}$$

Evans (Theorems 2.1 and 3.1 in [5]) proves that

$$\lim_{k\to+\infty}\int_{\mathbb{T}^n} H(\tilde{I}+\frac{\partial u_k}{\partial\varphi},\varphi)\sigma_k(\tilde{I},\varphi)d\varphi = \bar{H}(\tilde{I}) = \lim_{k\to+\infty}\bar{H}_k(\tilde{I}), \tag{10}$$

[1] Precisely, H is periodic in the φ variables; H is convex in the I variables; there exists $C>0$ such that, for any $I\in\mathbb{R}^n$ and $\varphi\in\mathbb{T}^n$: $\max\left\{\left|\frac{\partial^2 H}{\partial I^2}\right|, \frac{\left|\frac{\partial^2 H}{\partial I\partial\varphi}\right|}{1+|I|}, \frac{\left|\frac{\partial^2 H}{\partial\varphi^2}\right|}{1+|I|^2}\right\}\leq C$.

where $\bar{H}$ is the usual effective Hamiltonian –see (4)– of weak KAM theory.

Since the functions u_k are smooth, they may be used to generate canonical transformations $(I, \varphi) \mapsto (\tilde{I}, \tilde{\varphi})$ up to the inversion of[2]

$$I = U_I(\tilde{I}, \varphi), \quad U_I(\tilde{I}, \varphi) = \tilde{I} + \frac{\partial u_k}{\partial \varphi}(\tilde{I}, \varphi) \tag{11}$$

and

$$\tilde{\varphi} = U_\varphi(\tilde{I}, \varphi), \quad U_\varphi(\tilde{I}, \varphi) = \varphi + \frac{\partial u_k}{\partial I}(\tilde{I}, \varphi). \tag{12}$$

For every fixed $\tilde{I} \in \mathbb{R}^n$, Evans introduces the dynamics on the torus $\mathbb{T}^n$ by the differential equation

$$\dot{\varphi} = \frac{\partial H}{\partial I}\Big(\tilde{I} + \frac{\partial u_k}{\partial \varphi}(\tilde{I}, \varphi), \varphi\Big), \tag{13}$$

whose flow will be here denoted by $\mathcal{C}^t_{\tilde{I}}(\varphi)$. This torus flow $\mathcal{C}^t_{\tilde{I}}(\varphi)$ preserves the measure defined by $\sigma_k(\tilde{I}, \varphi)$, see [6]. Indeed, from the Euler-Lagrange equation related to the variation of $I_k[u]$, one obtains

$$\mathrm{Div}_\varphi\Big(\sigma_k(\tilde{I}, \varphi)\frac{\partial H}{\partial I}\Big(\tilde{I} + \frac{\partial u_k}{\partial \varphi}(\tilde{I}, \varphi), \varphi\Big)\Big) = 0. \tag{14}$$

Starting from $\mathcal{C}^t_{\tilde{I}}(\varphi)$, and inspired by equations (2.16)-(2.18) of [6], we define the following two evolutions

$$\Phi^t : (\tilde{I}, \varphi) \to (I^t, \varphi^t) := (U_I(\tilde{I}, \mathcal{C}^t_{\tilde{I}}(\varphi)), \mathcal{C}^t_{\tilde{I}}(\varphi)) \tag{15}$$

and

$$\tilde{\Phi}^t : (\tilde{I}, \varphi) \to (\tilde{I}^t, \tilde{\varphi}^t) := (\tilde{I}, U_\varphi(\tilde{I}, \mathcal{C}^t_{\tilde{I}}(\varphi))), \tag{16}$$

which are obtained as the composition of the flow

$$\tau^t : (\tilde{I}, \varphi) \to (\tilde{I}, \mathcal{C}^t_{\tilde{I}}(\varphi))$$

and the transformations (11), (12) of actions and angles respectively (see Figure 1). We remark that the (I^t, φ^t) and $(\tilde{I}^t, \tilde{\varphi}^t)$ are not necessarily conjugate, since expressions (11), (12) are not necessarily invertible.

2.1 Relations between the torus dynamics and the Hamiltonian flows

We stress that the very dynamic relevance of the $\tilde{I}$-collection of flows $\mathcal{C}^t_{\tilde{I}}(\varphi)$ lies in the relation between the orbits of (15), (16) and those of the Hamiltonian flows

$$\Phi^t_H : (I, \varphi) \to (I^t_H, \varphi^t_H) \qquad \Phi^t_{\bar{H}_k} : (\tilde{I}, \tilde{\varphi}) \to (\tilde{I}^t_{\bar{H}_k}, \tilde{\varphi}^t_{\bar{H}_k}) := (\tilde{I}, \tilde{\varphi} + t\frac{\partial \bar{H}_k}{\partial I}(\tilde{I})) \tag{17}$$

[2]Of course, the functions U_I, U_φ depend on the parameter k. But, since we will not define their limit for k tending to infinity, we prefer to use this simplified implicit notation.

$$
\begin{tikzcd}
& (I^t, \varphi^t) \\
(\tilde{I}, \varphi) \arrow[ur, "\Phi^t"] \arrow[r, "\tau"] \arrow[dr, "\tilde{\Phi}^t"'] & (\tilde{I}, \mathcal{C}^t_{\tilde{I}}(\varphi)) \arrow[u, "actions\ lift"'] \arrow[d, "angles\ lift"] \\
& (\tilde{I}^t, \tilde{\varphi}^t)
\end{tikzcd}
$$

Figure 1: The dynamics (15) and (16) represent different lifts of the torus flow dynamics. We note that in both cases the domain is the mixed variables set $\mathbb{R}^n \times \mathbb{T}^n$.

$$
\begin{tikzcd}
(I^t_H, \varphi^t_H) \arrow[r, dotted, no head, "(\star)"] & (I^t, \varphi^t) & \\
(I, \varphi) \arrow[u, "\Phi^t_H"] & (\tilde{I}, \varphi) \arrow[l, dashed, "U_I \times\ id"] \arrow[u, "\Phi^t"'] \arrow[r, dashed, "id\ \times U_\varphi"] \arrow[d, "\tilde{\Phi}^t"'] & (\tilde{I}, \tilde{\varphi}) \arrow[d, "\Phi^t_{\bar{H}_k}"] \\
& (\tilde{I}^t, \tilde{\varphi}^t) \arrow[r, dotted, no head, "(\bullet)"'] & (\tilde{I}^t_{\bar{H}_k}, \tilde{\varphi}^t_{\bar{H}_k})
\end{tikzcd}
$$

Figure 2: The dynamics are represented by solid arrows; the transformations of variables by dashed arrows; dotted lines link the evolutions (15), (16) and the Hamiltonian flows (17).

of H and $\bar{H}_k$ (defined in (9)) respectively.

In [6] the relation between $\tilde{\Phi}^t$ and the Hamiltonian flow $\Phi^t_{\bar{H}_k}$ (($\bullet$) in Figure 2) is expressed through the asymptotic formula (2.22). Specifically, Evans proves that there exists a constant $C_R > 0$ such that

$$E_1(k) := \int_{|\tilde{I}|\le R} \int_{\mathbb{T}^n} \Big| \frac{\partial \tilde{\varphi}^t}{\partial t} - \frac{\partial \bar{H}_k}{\partial I}(\tilde{I}) \Big|^2 \sigma_k(\tilde{I}, \varphi) d\varphi d\tilde{I} \le \frac{C_R}{k} \tag{18}$$

$\forall t \in \mathbb{R}$. Moreover he shows (formula (2.21) in [6]) that for some $C^R > 0$

$$E_2(k) := \int_{\mathbb{T}^n} \Big| \frac{\partial I^t}{\partial t} + \frac{\partial H}{\partial \varphi}(\Phi^t(\tilde{I}, \varphi)) \Big|^2 \sigma_k(\tilde{I}, \varphi) d\varphi \le \frac{C^R}{k} \tag{19}$$

$\forall t \in \mathbb{R}$ and $|\tilde{I}| \le R$.

By the next proposition, we complete the dynamic picture by studying the relation ($\star$) in Figure 2. More precisely, using the estimate (19) of Evans, we measure the gap

$$d_k(t, \tilde{I}, \varphi) := |\Phi^t(\tilde{I}, \varphi) - \Phi^t_H(U_I(\tilde{I}, \varphi), \varphi)| = |(I^t - I^t_H, \varphi^t - \varphi^t_H)| \tag{20}$$

between the curves (I^t_H, φ^t_H) and (I^t, φ^t) in terms of σ_k.

Proposition 2.1 *Let $\lambda_H > 0$ be a Lipschitz constant for the Hamiltonian vector field of X_H. Then, we have*

$$\int_{\mathbb{T}^n} d_k(t,\tilde{I},\varphi)\sigma_k(\tilde{I},\varphi)d\varphi \leq \frac{(e^{\lambda_H t}-1)(1+C^R)}{\lambda_H\sqrt{k}} \tag{21}$$

$\forall t \in \mathbb{R}$, $k \in \mathbb{N}$ and $|\tilde{I}| \leq R$. In particular,

$$\lim_{k\to+\infty}\int_{\mathbb{T}^n} d_k(t,\tilde{I},\varphi)\sigma_k(\tilde{I},\varphi)d\varphi = 0$$

$\forall t \in \mathbb{R}$ and $|\tilde{I}| \leq R$.

We remark that the presence of the exponential term $e^{\lambda_H t}$ in (21) is not surprising, since any small correction to a differential equation typically produces an exponential divergence of the solutions. In Section 7 we provide some further detail regarding this divergence, by considering an example with exponential divergence due to the presence of a hyperbolic equilibrium point of the Hamiltonian flow.

The proof of the proposition is based on the next technical

Lemma 2.2 *For any $t \in \mathbb{R}$, $k \in \mathbb{N}$, $\tilde{I} \in \mathbb{R}^n$ and $\varphi \in \mathbb{T}^n$, we have*

$$d_k(t,\tilde{I},\varphi) \leq e^{\lambda_H t}\int_0^t \left|\frac{\partial I^t}{\partial t}\Big|_{t=s} + \frac{\partial H}{\partial \varphi}(\Phi^s(\tilde{I},\varphi))\right| e^{-\lambda_H s} ds. \tag{22}$$

Proof of Lemma. The time derivatives of I^t, φ^t satisfy (see (15))

$$\begin{aligned}
\frac{\partial I^t}{\partial t} &= \frac{\partial^2 u_k}{\partial \varphi^2}(\tilde{I},\varphi^t)\frac{\partial H}{\partial I}(U_I(\tilde{I},\varphi^t),\varphi^t) = \frac{\partial^2 u_k}{\partial \varphi^2}(\tilde{I},\varphi^t)\frac{\partial H}{\partial I}(I^t,\varphi^t),\\
\frac{\partial \varphi^t}{\partial t} &= \frac{\partial H}{\partial I}(U_I(\tilde{I},\varphi^t),\varphi^t) = \frac{\partial H}{\partial I}(I^t,\varphi^t)
\end{aligned}$$

so that the functions I^t, φ^t may be interpreted as the solutions of the following $\tilde{I}$-parametric differential equation

$$\begin{cases} \dot{I} = \frac{\partial^2 u_k}{\partial \varphi^2}(\tilde{I},\varphi)\frac{\partial H}{\partial I}(I,\varphi) \\ \dot{\varphi} = \frac{\partial H}{\partial I}(I,\varphi) \end{cases}$$

with special initial conditions $(I,\varphi) = (U_I(\tilde{I},\varphi),\varphi)$, and solutions denoted by $\Phi^t(\tilde{I},\varphi)$, see (15).
Let

$$X_{\tilde{I}}(I,\varphi) = \Big(\frac{\partial^2 u_k}{\partial \varphi^2}(\tilde{I},\varphi)\frac{\partial H}{\partial I}(I,\varphi), \frac{\partial H}{\partial I}(I,\varphi)\Big)$$

and

$$X_H(I,\varphi) = \Big(-\frac{\partial H}{\partial \varphi}(I,\varphi), \frac{\partial H}{\partial I}(I,\varphi)\Big).$$

Since

$$d_k(t,\tilde{I},\varphi) = |(I^t - I^t_H, \varphi^t - \varphi^t_H)|,$$

its time derivative

$$\frac{\partial d_k}{\partial t} = \frac{1}{d_k}\Big(I^t - I^t_H, \varphi^t - \varphi^t_H\Big) \cdot \Big(X_{\tilde I}(I^t,\varphi^t) - X_H(I^t_H,\varphi^t_H)\Big)$$

is well defined only for $d_k(t,\tilde I,\varphi) > 0$, for example, $d_k(0,\tilde I,\varphi) = 0$. In order to overcome the lack of differentiability, for a constant $\varepsilon > 0$ we consider the function

$$x_k(t,\tilde I,\varphi) := \sqrt{d_k(t,\tilde I,\varphi)^2 + \varepsilon^2},$$

whose time derivative

$$\frac{\partial x_k}{\partial t} = \frac{1}{x_k}\Big(I^t - I^t_H, \varphi^t - \varphi^t_H\Big) \cdot \Big(X_{\tilde I}(I^t,\varphi^t) - X_H(I^t_H,\varphi^t_H)\Big)$$

satisfies

$$\begin{aligned}\left|\tfrac{\partial x_k}{\partial t}\right| &\le \tfrac{d_k}{x_k}\left|X_{\tilde I}(I^t,\varphi^t) - X_H(I^t_H,\varphi^t_H)\right| \le \left|X_{\tilde I}(I^t,\varphi^t) - X_H(I^t_H,\varphi^t_H)\right| \le \\ &\le \left|X_{\tilde I}(I^t,\varphi^t) - X_H(I^t,\varphi^t)\right| + \left|X_H(I^t,\varphi^t) - X_H(I^t_H,\varphi^t_H)\right|.\end{aligned}$$

As a consequence, we have ($d_k < x_k$)

$$\left|\frac{\partial x_k}{\partial t}\right| \le \left|X_{\tilde I}(I^t,\varphi^t) - X_H(I^t,\varphi^t)\right| + \lambda_H d_k < \left|X_{\tilde I}(I^t,\varphi^t) - X_H(I^t,\varphi^t)\right| + \lambda_H x_k,$$

where $\lambda_H > 0$ is a Lipschitz constant for X_H. By recalling now the classical *a priori* estimate[3], we obtain

$$x_k(t,\tilde I,\varphi) \le \int_0^t \left|X_{\tilde I}(I^s,\varphi^s) - X_H(I^s,\varphi^s)\right| e^{\lambda_H(t-s)} ds + e^{\lambda_H t} x_k(0),$$

with $x_k(0) = \varepsilon$, so that

$$d_k(t,\tilde I,\varphi) \le e^{\lambda_H t}\Big(\int_0^t \left|X_{\tilde I}(I^s,\varphi^s) - X_H(I^s,\varphi^s)\right| e^{-\lambda_H s} ds + \varepsilon\Big).$$

As a consequence, by considering arbitrarily small $\varepsilon > 0$, we conclude that

$$d_k(t,\tilde I,\varphi) \le e^{\lambda_H t}\int_0^t \left|X_{\tilde I}(I^s,\varphi^s) - X_H(I^s,\varphi^s)\right| e^{-\lambda_H s} ds.$$

Finally, since

$$X_{\tilde I}(I^s,\varphi^s) - X_H(I^s,\varphi^s) = \Big(\frac{\partial I^t}{\partial t}\Big|_{t=s} + \frac{\partial H}{\partial \varphi}(I^s,\varphi^s), 0\Big) = \Big(\frac{\partial I^t}{\partial t}\Big|_{t=s} + \frac{\partial H}{\partial \varphi}(\Phi^s(\tilde I,\varphi)), 0\Big),$$

we have

$$d_k(t,\tilde I,\varphi) \le e^{\lambda_H t}\int_0^t \left|\frac{\partial I^t}{\partial t}\Big|_{t=s} + \frac{\partial H}{\partial \varphi}(\Phi^s(\tilde I,\varphi))\right| e^{-\lambda_H s} ds$$

[3] In brief, the *a priori* upper bound estimate lemma, see [19]: if $|f'(t)| < M(t,|f(t)|)$ and $g(t)$ solves $\dot g(t) = M(t,g(t))$ with $g(0) = f(0)$, then $|f(t)| \le g(t)$.

for any $t \in \mathbb{R}$, $k \in \mathbb{N}$, $\tilde{I} \in \mathbb{R}^n$ and $\varphi \in \mathbb{T}^n$. □

Proof of Proposition 2.1. By considering the σ_k-average of d_k and the inequality (22), we obtain

$$\int_{\mathbb{T}^n} d_k(t, \tilde{I}, \varphi)\sigma_k(\tilde{I}, \varphi)d\varphi \leq e^{\lambda_H t} \int_0^t e^{-\lambda_H s} \int_{\mathbb{T}^n} \left| \frac{\partial I^t}{\partial t}\Big|_{t=s} + \frac{\partial H}{\partial \varphi}(\Phi^s(\tilde{I}, \varphi)) \right| \sigma_k(\tilde{I}, \varphi)d\varphi ds.$$

Since now for any $x \geq 0$ and $k \in \mathbb{N}$

$$x \leq \max\left(\frac{1}{\sqrt{k}}, \sqrt{k}\, x^2\right) \leq \frac{1}{\sqrt{k}} + \sqrt{k}\, x^2,$$

we conclude that

$$\begin{aligned} \int_{\mathbb{T}^n} \left| \frac{\partial I^t}{\partial t}\Big|_{t=s} + \frac{\partial H}{\partial \varphi}(\Phi^s(\tilde{I}, \varphi)) \right| \sigma_k(\tilde{I}, \varphi)d\varphi &\leq \int_{\mathbb{T}^n} \left(\frac{1}{\sqrt{k}} + \sqrt{k} \left| \frac{\partial I^t}{\partial t}\Big|_{t=s} + \frac{\partial H}{\partial \varphi}(\Phi^s(\tilde{I}, \varphi)) \right|^2 \right) \sigma_k(\tilde{I}, \varphi)d\varphi \\ &\leq \frac{1}{\sqrt{k}} + \frac{C^R}{\sqrt{k}}, \end{aligned}$$

where, in the last inequality, we have used the estimate (19) of Evans. Since the right hand side of the inequality does not depend on s, we immediately obtain (21). □

3 Explicit solutions and convergences in the one dimensional case

This section presents the explicit formula for the minimizers u_k of the functional $I_k[u]$ defined in (7) for the Hamiltonian systems (5) with one degree of freedom.

For simplicity we consider only the action interval $\tilde{I} > 0$: the case $\tilde{I} < 0$ can be obtained by symmetry (see Remark **(IV)** below). In the sequel we make extensive use of the Lambert function W, defined implicitly by $z = W(z)e^{W(z)}$, and also its asymptotic properties. (We refer the reader to the technical Section 8 and to [3],[4]).

Definition 3.1 *For $H(I, \varphi) = I^2/2 + f(\varphi) \in \mathcal{C}^2(\mathbb{R} \times \mathbb{S}^1)$, let us define:*

(*i*) *For $\tilde{I} > 0$, the sequence of functions $c_k(\tilde{I}) \in \mathbb{R}$ by inversion of*

$$c \longmapsto \tilde{I} = \frac{1}{2\pi} \int_0^{2\pi} \gamma_k(c, \varphi)d\varphi, \tag{23}$$

where

$$\gamma_k(c, \varphi) := \sqrt{\frac{W(e^{2(c-f(\varphi))k}k)}{k}}. \tag{24}$$

(*ii*) *For $\tilde{I} > 0$, the function $c(\tilde{I}) \in \mathbb{R}$ by inversion of*

$$c \longmapsto \tilde{I} = \frac{1}{2\pi} \int_0^{2\pi} \gamma_0(c, \varphi)d\varphi, \tag{25}$$

where

$$\gamma_0(c, \varphi) := \begin{cases} \sqrt{2(c - f(\varphi))} & \text{if } c > f(\varphi) \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

(iii) *For* $(\tilde{I},\varphi)\in(0,+\infty)\times\mathbb{S}^1$, *the sequences of functions*

$$u_k(\tilde{I},\varphi) := \tilde{I}(\pi-\varphi) - \frac{1}{2\pi}\int_0^{2\pi}\int_0^y \gamma_k(c_k(\tilde{I}),x)dxdy + \int_0^{\varphi}\gamma_k(c_k(\tilde{I}),x)dx \tag{27}$$

and

$$\sigma_k(\tilde{I},\varphi) := \frac{\frac{1}{\gamma_k(c_k(\tilde{I}),\varphi)}}{\int_0^{2\pi}\frac{1}{\gamma_k(c_k(\tilde{I}),x)}dx}. \tag{28}$$

(iv) *For* $(\tilde{I},\varphi)\in(0,+\infty)\times\mathbb{S}^1$, *the function*

$$u_0(\tilde{I},\varphi) := \tilde{I}(\pi-\varphi) - \frac{1}{2\pi}\int_0^{2\pi}\int_0^y \gamma_0(c(\tilde{I}),x)dxdy + \int_0^{\varphi}\gamma_0(c(\tilde{I}),x)dx \tag{29}$$

(v) *For* $\tilde{I}>0$, *the sequence of functions*

$$\bar{H}_k(\tilde{I}) := c_k(\tilde{I}) + \frac{1}{k}\log\int_0^{2\pi}\frac{1}{\gamma_k(c_k(\tilde{I}),\varphi)}d\varphi. \tag{30}$$

The convergence properties of the objects defined above are stated in the following

Theorem 3.2 *Let us consider* $H(I,\varphi)=I^2/2+f(\varphi)\in\mathcal{C}^2(\mathbb{R}\times\mathbb{S}^1)$.

(i) *For any* $\tilde{I}>0$, *the functions* $u_k(\tilde{I},\varphi)$ *defined in (27) are smooth, have zero average and solve the Euler-Lagrange equation (14) for* $I_k[u]$. *Moreover, any* $u_k(\tilde{I},\varphi)$ *converges uniformly to* $u_0(\tilde{I},\varphi)$ *on* $\mathbb{S}^1$.

(ii) *The functions* $\gamma_k(c,\varphi)$ *defined in (24) are smooth and uniformly converging to* $\gamma_0(c,\varphi)$ *defined in (26) on* $(-\infty,c_*]\times\mathbb{S}^1$, *for any fixed* $c_*\in\mathbb{R}$.

(iii) *The functions* $c_k(\tilde{I})$ *are pointwise converging to* $c(\tilde{I})$.

(iv) *The functions* $\sigma_k(\tilde{I},\varphi)$ *and* $\bar{H}_k(\tilde{I})$, *defined in (28) and (30) respectively, satisfy*

$$\lim_{k\to+\infty}\int_0^{2\pi} H(\tilde{I}+\frac{\partial u_k}{\partial\varphi},\varphi)\sigma_k(\tilde{I},\varphi)d\varphi = \bar{H}(\tilde{I}) = \lim_{k\to+\infty}\bar{H}_k(\tilde{I}), \tag{31}$$

where

$$\bar{H}(\tilde{I}) := \begin{cases} c(\tilde{I}) & \text{if } c(\tilde{I})>\max f \\ \max f & \text{otherwise}\end{cases} \tag{32}$$

Remarks

(I) As we will see in Section 5.1, the functions $\gamma_k(c,\varphi)$ parametrize the level curves for the Hamiltonians (6) of value c and are well defined for any $\varphi\in\mathbb{S}^1$. In other words, these level curves project

injectively on $\mathbb{S}^1$. Therefore, the action $\tilde{I}$ in (23) is proportional to the area of the phase-space $(0, +\infty) \times \mathbb{S}^1$ under the graph of $\gamma_k(c_k(\tilde{I}), \varphi)$. More precisely, we have

$$\tilde{I} = \frac{1}{2\pi} \int_0^{2\pi} \gamma_k(c_k(\tilde{I}), \varphi) d\varphi \tag{33}$$

as well as

$$\tilde{I} = \frac{1}{2\pi} \int_0^{2\pi} \gamma_0(c(\tilde{I}), \varphi) d\varphi. \tag{34}$$

Let us remark that $\tilde{I} > 0$ corresponds to $c(\tilde{I}) > \min f$.

(II) The functions $u_k(\tilde{I}, \varphi)$ are smooth solutions of the Hamilton-Jacobi equation for (6),

$$\frac{1}{2}(\tilde{I} + \frac{\partial u_k}{\partial \varphi})^2 + f(\varphi) + \frac{1}{k} \log(\tilde{I} + \frac{\partial u_k}{\partial \varphi}) = c \tag{35}$$

with $c = c_k(\tilde{I})$. The PDE (35), at variance with the Hamilton-Jacobi equation for (5), admits smooth solutions defined over $\mathbb{S}^1$ for all values $c > 0$. Once again, this follows because all level curves of (35) project injectively on $\mathbb{S}^1$.
Let us also remark that, while in the general n dimensional setting Evans ([5], Lemma 2.1) assumes the uniform convergence of the sequence u_k, passing if necessary to a subsequence, in the one dimensional case we can prove the stronger uniform convergence of u_k to u_0 on $\mathbb{S}^1$.

(III) For $\tilde{I} > 0$ such that $c(\tilde{I}) > \max f$, the function $\varphi \mapsto \tilde{I}\varphi + u_0(\tilde{I}, \varphi)$ provides a regular solution to the Hamilton-Jacobi equation for the Hamiltonian H, see (5), on the energy level $c(\tilde{I})$. Notice that $I\varphi + u_0(\tilde{I}, \varphi)$ represents the generating function conjugating H to $\bar{H} = c(\tilde{I})$. Otherwise, for $c(\tilde{I}) \leq \max f$ the picture differs from the classical integration of one dimensional Hamiltonian systems, because $\gamma_0(c(\tilde{I}), \varphi)$ has angular points for $c(\tilde{I}) = f(\varphi)$ and the limit function $u_0(\tilde{I}, \varphi)$ is therefore only Lipschitz.

(IV) The case $\tilde{I} < 0$ is obtained via the choice

$$u_k(\tilde{I}, \varphi) := \tilde{I}(\pi - \varphi) + \frac{1}{2\pi} \int_0^{2\pi} \int_0^y \gamma_k(c_k(|\tilde{I}|), x) dx dy - \int_0^{\varphi} \gamma_k(c_k(|\tilde{I}|), x) dx.$$

As a consequence one also has $\bar{H}_k(\tilde{I}) = \bar{H}_k(|\tilde{I}|)$ and $\sigma_k(\tilde{I}, \varphi) = \sigma_k(|\tilde{I}|, \varphi)$.

We devote here some attention to the convergence properties of the density measures σ_k. In the generic n dimensional case, Evans [5, 6] discusses the consequences of the convergence

$$\sigma_k \rightharpoonup \sigma \qquad \text{weakly as measures on } \mathbb{T}^n$$

possibly through a sub-sequence.

A particularly interesting case corresponds to the convergence of the σ_k to singular measures on the torus $\mathbb{T}^n$. Unfortunately, the theory of [5] and [6] does not provide explicit examples. In the one dimensional case, if $c(\tilde{I}) > \max f$, the limit of σ_k obviously defines a regular measure on $\mathbb{S}^1$. The case $c(\tilde{I}) \leq \max f$ is actually more tricky to manage. The following proposition gives an example of convergence to a singular measure:

Proposition 3.3 *Let* $f(\varphi) = -\cos\varphi$ *and* $\tilde{I} > 0$ *be such that* $c(\tilde{I}) = f(\pi) = 1$. *For any test function* $u \in C^\infty([0, 2\pi); \mathbb{R})$, *one has*

$$\lim_{k\to+\infty} \int_0^{2\pi} u(\varphi)\sigma_k(\tilde{I}, \varphi)d\varphi = u(\pi).$$

4 Lower bounds and outcomes in the n dimensional case

By exploiting our explicit knowledge of the sequences u_k and σ_k in the one dimensional case, we first propose to give refined asymptotic estimates for the integrals (18) and (19). The estimates are also relevant for the n dimensional case. We start with the following:

Theorem 4.1 *Let us consider* $H(I, \varphi) = I^2/2 + f(\varphi) \in \mathcal{C}^2(\mathbb{R} \times \mathbb{S}^1)$, *where* f *is a non constant function.*

(*i*) *For any* $r > 0$ *and* $R > 0$ *satisfying* $c(R) > \max f + r$, *there exist* $K > 0$ *and* $c_R > 0$ *such that*

$$E_1(k) \geq \frac{c_R}{k^2} \tag{36}$$

$\forall k > K$.

(*ii*) *For any* $\tilde{I} > 0$ *such that* $c(\tilde{I}) > \max f$, *there exists* $c_{\tilde{I}} > 0$ *such that*

$$\lim_{k\to+\infty} k^2 E_2(k) = c_{\tilde{I}}. \tag{37}$$

In particular, we have

$$c_R := 4\pi \int_{\max f + r}^{c(R)} \frac{-a^2_{\frac{3}{2}}(c) + a_{\frac{5}{2}}(c)a_{\frac{1}{2}}(c)}{a^3_{\frac{1}{2}}(c)} dc$$

and

$$c_{\tilde{I}} := \frac{1}{a_{\frac{1}{2}(c(\tilde{I}))}} \int_0^{2\pi} \frac{|f'(\varphi)|^2}{\gamma_0^5(c(\tilde{I}), \varphi)} d\varphi = \frac{1}{a_{\frac{1}{2}(c(\tilde{I}))}} \int_0^{2\pi} \frac{|f'(\varphi)|^2}{[2(c(\tilde{I}) - f(\varphi))]^{5/2}} d\varphi,$$

where

$$a_\delta(c) := \int_0^{2\pi} \frac{1}{\gamma_0^{2\delta}(c, \varphi)} d\varphi = \int_0^{2\pi} \frac{1}{[2(c - f(\varphi))]^\delta} d\varphi.$$

Theorem 4.1 provides lower bounds which are also significant for the generic n dimensional case. Indeed, for the n degrees of freedom mechanical Hamiltonians

$$H(I_1, \ldots, I_n, \varphi_1, \ldots, \varphi_n) = \sum_{j=1}^n \frac{I_j^2}{2} + f(\varphi_1, \ldots, \varphi_n), \tag{38}$$

let us consider Evans construction of the sequences $u_k^{(n)}$, $\bar{H}_k^{(n)}$ and $\sigma_k^{(n)}$, as well as the integrals $E_1(k), E_2(k)$. A relevant question is that regarding optimality of the upper bounds (18) and (19) proved in Evans paper.

Of course, in the trivial integrable case $H(I) = I^2/2$, both $E_1(k)$ and $E_2(k)$ are zero. However, already in the quadrature-integrable case, for example

$$f(\varphi_1, \dots, \varphi_n) := f(\varphi_1),$$

we have

$$\frac{c_R}{k^2} \le E_1(k) \le \frac{C_R}{k} \qquad \text{and} \qquad \frac{c_{\tilde{I}}}{k^2} \le E_2(k) \le \frac{C^R}{k} \tag{39}$$

for any $c(R) > \max f$ (see Theorem 4.1) and sufficiently large k. In fact all the sequences can be constructed by referring to the one dimensional case

$$u_k^{(n)}(\tilde{I}_1, \dots, \tilde{I}_n, \varphi_1, \dots, \varphi_n) := u_k^{(1)}(\tilde{I}_1, \varphi_1), \qquad \sigma_k^{(n)}(\tilde{I}_1, \dots, \tilde{I}_n, \varphi_1, \dots, \varphi_n) := \sigma_k^{(1)}(\tilde{I}_1, \varphi_1)$$

and

$$\bar{H}_k^{(n)}(\tilde{I}_1, \dots, \tilde{I}_n) = \bar{H}_k^{(1)}(\tilde{I}_1) + \sum_{j=2}^{n} \frac{\tilde{I}_j^2}{2}.$$

Moreover, with regard to the mechanical Hamiltonian systems (38), the integrals in (18) and (19) can be written precisely in the following form (see Theorem 4.1):

$$E_1(k) = \int_{|\tilde{I}| \le R} \int_{\mathbb{T}^n} \left| \frac{\partial}{\partial I} \left[\frac{1}{2} \left(\tilde{I} + \frac{\partial u_k^{(n)}}{\partial \varphi} \right)^2 - \bar{H}_k^{(n)}(\tilde{I}) \right] \right|^2 \sigma_k^{(n)} d\varphi d\tilde{I}$$

and

$$E_2(k) = \int_{\mathbb{T}^n} \left| \frac{\partial}{\partial \varphi} \left[\frac{1}{2} \left(\tilde{I} + \frac{\partial u_k^{(n)}}{\partial \varphi} \right)^2 + f(\varphi) \right] \right|^2 \sigma_k^{(n)} d\varphi.$$

As a consequence, we immediately conclude that the estimates (39) also apply to the quadrature-integrable n dimensional case.

5 Proof of Theorem 3.2

5.1 Explicit formulas for u_k

According to [6], the function u_k is a minimizer of the functional $I_k[u]$ defined in (7), whose Euler-Lagrange equation is

$$\sum_{j=1}^{n} \frac{\partial}{\partial \varphi_j} \left(e^{kH(\tilde{I} + \frac{\partial u_k}{\partial \varphi}, \varphi)} \frac{\partial H}{\partial I}(\tilde{I} + \frac{\partial u_k}{\partial \varphi}, \varphi) \right) = 0.$$

In the special one dimensional case, the previous equation becomes

$$\frac{d}{d\varphi} \left(e^{kH(\tilde{I} + \frac{\partial u_k}{\partial \varphi}, \varphi)} \frac{\partial H}{\partial I}(\tilde{I} + \frac{\partial u_k}{\partial \varphi}, \varphi) \right) = 0.$$

This can be integrated and one obtains

$$e^{kH(\tilde{I} + \frac{\partial u_k}{\partial \varphi}, \varphi)} \left(\frac{\partial H}{\partial I}(\tilde{I} + \frac{\partial u_k}{\partial \varphi}, \varphi) \right) = c \tag{40}$$

for some $c \in \mathbb{R}$. For $H(I,\varphi) = I^2/2 + f(\varphi)$, we have

$$e^{kH(\tilde{I}+\frac{\partial u_k}{\partial \varphi},\varphi)}\left(\tilde{I} + \frac{\partial u_k}{\partial \varphi}\right) = c. \tag{41}$$

From equation (41) one immediately recognizes that the constant c has the same sign as $\tilde{I} + \frac{\partial u_k}{\partial \varphi}$ and $\tilde{I}$. It suffices to first write (41) as $\tilde{I} + \frac{\partial u_k}{\partial \varphi} = ce^{-kH(\tilde{I}+\frac{\partial u_k}{\partial \varphi},\varphi)}$, and then to average both sides over φ. Therefore, for $\tilde{I} > 0$, $c > 0$ one also has $\tilde{I} + \frac{\partial u_k}{\partial \varphi} > 0$ for any $\varphi \in \mathbb{S}^1$, so that we can write equation (41) in the form

$$e^{k\left(\frac{1}{2}(\tilde{I}+\frac{\partial u_k}{\partial \varphi})^2+f(\varphi)\right)+\log(\tilde{I}+\frac{\partial u_k}{\partial \varphi})} = c.$$

Thus, on putting $c_k := \frac{\log c}{k}$, one has

$$\frac{1}{2}(\tilde{I} + \frac{\partial u_k}{\partial \varphi})^2 + \frac{1}{k}\log(\tilde{I} + \frac{\partial u_k}{\partial \varphi}) + f(\varphi) = c_k. \tag{42}$$

Bearing in mind the Lambert function W (see Section 8), we see that the equation $\frac{1}{2}\gamma_k^2 + \frac{1}{k}\log(\gamma_k) + f(\varphi) = c_k$ may be written in the form

$$e^{k\gamma_k^2}(k\gamma_k^2) = e^{2k(c_k - f(\varphi))}k,$$

and since the right hand side is positive, we can represent its solution –see formula (79)– as

$$k\gamma_k^2 = W(e^{2(c_k - f(\varphi))k}k), \tag{43}$$

that is,

$$\tilde{I} + \frac{\partial u_k}{\partial \varphi} = \gamma_k(c_k,\varphi) = \sqrt{\frac{W(e^{2(c_k-f(\varphi))k}k)}{k}}. \tag{44}$$

Integrating (44) over $[0,\varphi]$ we find

$$u_k(\tilde{I},\varphi) = u_k(\tilde{I},0) - \tilde{I}\varphi + \int_0^{\varphi} \gamma_k(c_k,x)dx.$$

If we now require that u_k be periodic with respect to φ, we have

$$\tilde{I} = \frac{1}{2\pi}\int_0^{2\pi} \gamma_k(c_k,\varphi)d\varphi,$$

while a function with zero average is obtained if one chooses

$$u_k(\tilde{I},0) = \tilde{I}\pi - \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\varphi} \gamma_k(c_k,x)dxd\varphi.$$

We have therefore proved that the function $u_k(\tilde{I},\varphi)$ in (27) has zero average and solves the Euler-Lagrange equation (14) for $I_k[u]$.
From definitions (8) and (9), and since

$$e^{kH(\tilde{I}+\frac{\partial u_k}{\partial \varphi},\varphi)} = e^{k(c_k - \frac{1}{k}\log\gamma_k)} = \frac{e^{kc_k}}{\gamma_k},$$

we immediately obtain (28) and (30). The limit (31) follows directly from [6] (specifically from (2.5) in [6], see also Theorems 2.1 and 3.1 in [5]), while structure (32) comes from the well known representation of the effective Hamiltonian for one dimensional systems. □

5.2 Uniform convergence of γ_k to γ_0

This section is devoted to proving the uniform convergence of $\gamma_k(c,\varphi)$ to $\gamma_0(c,\varphi)$ on compact sets of $\mathbb{R}\times\mathbb{S}^1$. Specifically, we prove that for any $\varepsilon>0$ and $c_*\in\mathbb{R}$, there exists $K(\varepsilon,c_*)$ such that, for any $k\geq K(\varepsilon,c_*)$, we have

$$|\gamma_k(c,\varphi)-\gamma_0(c,\varphi)|\leq\varepsilon$$

$\forall(c,\varphi)\in(-\infty,c_*]\times\mathbb{S}^1$. This result will be essential in the proof of the pointwise convergence of c_k to c. We distinguish two different cases (i) and (ii).

(i) Let us consider φ such that $c\geq f(\varphi)$. We start with the following

Lemma 5.1 *Let $c\in\mathbb{R}$. For any $\varepsilon>0$ there exists $K_0(\varepsilon)$ independent of c such that, for any $k>K_0(\varepsilon)$ and φ satisfying $c\geq f(\varphi)$, we have*

$$\frac{\left|\gamma_k(c,\varphi)-\sqrt{2(c-f(\varphi))+\frac{\log k}{k}}\right|}{\sqrt{2(c-f(\varphi))+\frac{\log k}{k}}}\leq\varepsilon. \tag{45}$$

Proof. From (80) we know that for any $\varepsilon>0$ there exists $K_0(\varepsilon)$ such that, for any $z>K_0(\varepsilon)$, we have

$$\left|\sqrt{\frac{W(z)}{\log z}}-1\right|\leq\varepsilon.$$

Moreover, since $c\geq f(\varphi)$ one also has

$$e^{2(c-f(\varphi))k}k\geq k.$$

As a consequence of the last two facts, for any $\varepsilon>0$ and $k>K_0(\varepsilon)$, we have

$$\left|\sqrt{\frac{W(e^{2(c-f(\varphi))k}k)}{2(c-f(\varphi))k+\log k}}-1\right|\leq\varepsilon$$

$\forall\varphi$ such that $c\geq f(\varphi)$. We write the above inequality as

$$\frac{\left|\sqrt{\frac{W(e^{2(c-f(\varphi))k}k)}{k}}-\sqrt{2(c-f(\varphi))+\frac{\log k}{k}}\right|}{\sqrt{2(c-f(\varphi))+\frac{\log k}{k}}}\leq\varepsilon,$$

from which the lemma immediately follows. □

The uniform convergence of $\gamma_k(c,\varphi)$ to $\gamma_0(c,\varphi)$ on the set $c\geq f(\varphi)$ is now a direct consequence of the lemma. If $c<\min f$ there is nothing to prove, since this set is empty. We can therefore assume

$c_* \geq c \geq \min f$. For any $\eta > 0$, from Lemma 5.1 there exists $K_0(\eta)$ such that, for any $k \geq K_0(\eta)$, we have

$$|\gamma_k(c,\varphi) - \gamma_0(c,\varphi)| \leq \eta\sqrt{2(c - f(\varphi)) + \frac{\log k}{k}} + \left|\sqrt{2(c - f(\varphi)) + \frac{\log k}{k}} - \sqrt{2(c - f(\varphi))}\right|.$$

Choosing $K_1(\eta)$ such that $\frac{\log k}{k} \leq \eta$ for any $k \geq K_1(\eta)$, we immediately obtain

$$|\gamma_k(c,\varphi) - \gamma_0(c,\varphi)| \leq \eta\sqrt{2(c - \min f) + \eta} + \sqrt{\eta} \leq \eta\sqrt{2(c_* - \min f) + \eta} + \sqrt{\eta} \tag{46}$$

$\forall k \geq \max\{K_0(\eta), K_1(\eta)\}$. Therefore, if for any $\varepsilon > 0$ we choose $\eta := \eta(\varepsilon, c_*)$ such that

$$\eta(\varepsilon, c_*)\sqrt{2(c_* - \min f) + \eta(\varepsilon, c_*)} + \sqrt{\eta(\varepsilon, c_*)} = \varepsilon,$$

we find that, for any $k \geq \max\{K_0(\eta(\varepsilon, c_*)), K_1(\eta(\varepsilon, c_*))\}$, one has

$$|\gamma_k(c,\varphi) - \gamma_0(c,\varphi)| \leq \varepsilon.$$

(ii) We now consider φ such that $c < f(\varphi)$. In this case $\gamma_0(c,\varphi) = 0$ and therefore

$$|\gamma_k(c,\varphi) - \gamma_0(c,\varphi)| = \sqrt{\frac{W(e^{2(c-f(\varphi))k}k)}{k}}.$$

Since W is an increasing function of $z \in [0, +\infty)$ and $e^{2(c-f(\varphi))k}k \leq k$, we have

$$\sqrt{\frac{W(e^{2(c-f(\varphi))k}k)}{k}} \leq \sqrt{\frac{W(k)}{k}}.$$

From (80) we obtain

$$\lim_{k\to+\infty}\sqrt{\frac{W(k)}{\log k}} = 1,$$

that is, for any $\eta > 0$ there exists $K_0(\eta)$ such that, for every $k \geq K_0(\eta)$,

$$\left|\sqrt{\frac{W(k)}{\log k}} - 1\right| \leq \eta \quad \Longrightarrow \quad \left|\sqrt{W(k)} - \sqrt{\log k}\right| \leq \eta\sqrt{\log k},$$

from which we have

$$\sqrt{\frac{W(k)}{k}} \leq \frac{\sqrt{\log k} + \left|\sqrt{W(k)} - \sqrt{\log k}\right|}{\sqrt{k}} \leq \sqrt{\frac{\log k}{k}}(1 + \eta).$$

Therefore, on choosing $k \geq K_1(\eta)$, it follows that

$$|\gamma_k(c,\varphi) - \gamma_0(c,\varphi)| \leq \sqrt{\eta}(1 + \eta). \tag{47}$$

For any $\varepsilon > 0$, we choose $\eta := \tilde{\eta}(\varepsilon)$ such that $\sqrt{\tilde{\eta}(\varepsilon)}(1 + \tilde{\eta}(\varepsilon)) = \varepsilon$.
Thus, for any $k \geq \max\{K_1(\tilde{\eta}(\varepsilon)), K_0(\tilde{\eta}(\varepsilon))\}$, we have

$$|\gamma_k(c,\varphi) - \gamma_0(c,\varphi)| \leq \varepsilon.$$

The uniform convergence is therefore proved by choosing

$$K(\varepsilon, c_*) := \max\{K_0(\eta(\varepsilon, c_*)), K_1(\eta(\varepsilon, c_*)), K_1(\tilde{\eta}(\varepsilon)), K_0(\tilde{\eta}(\varepsilon))\}.$$

5.3 Pointwise convergence of c_k to c

In this section we prove that, for any $\tilde{I} > 0$, we have

$$\lim_{k \to +\infty} c_k(\tilde{I}) = c(\tilde{I}).$$

The proof is structured into points $(i) - (iv)$.

(i) We first establish that the sequence $c_k(\tilde{I})$ is bounded from above.
On the contrary, let us suppose the existence of a diverging sub-sequence $c_{k_i}(\tilde{I})$:

$$\lim_{i \to \infty} c_{k_i}(\tilde{I}) = +\infty.$$

From the monotonicity of W, for any $\varphi \in \mathbb{S}^1$, we have

$$\sqrt{\frac{W(e^{2(c_{k_i}(\tilde{I})-f(\varphi))k_i}k_i)}{k_i}} \geq \sqrt{\frac{W(e^{2(c_{k_i}(\tilde{I})-\max f)k_i}k_i)}{k_i}},$$

so that, by integrating in $\varphi \in [0, 2\pi]$ and using (23) and (24), we obtain

$$\tilde{I} \geq \sqrt{\frac{W(e^{2(c_{k_i}(\tilde{I})-\max f)k_i}k_i)}{k_i}}. \tag{48}$$

Moreover, as a consequence of (80), the divergence of $c_{k_i}(\tilde{I})$ implies the divergence of the sequence $a_i = \sqrt{\frac{W(e^{2(c_{k_i}(\tilde{I})-\max f)k_i}k_i)}{k_i}}$. Indeed one has

$$\lim_{i \to +\infty} a_i^2 = \lim_{i \to +\infty} \frac{W(e^{2(c_{k_i}(\tilde{I})-\max f)k_i}k_i)}{2(c_{k_i}(\tilde{I}) - \max f)k_i + \log k_i}\Big(2(c_{k_i}(\tilde{I}) - \max f) + \frac{\log k_i}{k_i}\Big) = +\infty.$$

But this is in contradiction with inequality (48).

(ii) We proceed by proving that, for $\tilde{I} > 0$, there exists $K_2(\tilde{I})$ such that

$$c_k(\tilde{I}) > c(\tilde{I}/4) > \min f$$

$\forall k \geq K_2(\tilde{I})$.
Let us first prove that $c_k(\tilde{I}) > c(\tilde{I}/4)$ for sufficiently large k. Point (i) provides the existence of $c_*(\tilde{I})$ for which $\sup_k c_k(\tilde{I}) < c_*(\tilde{I})$. Therefore, from the uniform convergence of γ_k to γ_0 in $(-\infty, c_*(\tilde{I})] \times \mathbb{S}^1$, we find that for any $\varepsilon > 0$ there exists $K(\varepsilon, c_*(\tilde{I}))$ such that, for any $k \geq K(\varepsilon, c_*(\tilde{I}))$ and $c < c_*(\tilde{I})$, one has

$$\frac{1}{2\pi}\left|\int_0^{2\pi} \gamma_k(c,\varphi)d\varphi - \int_0^{2\pi} \gamma_0(c,\varphi)d\varphi\right| \leq \varepsilon.$$

In particular,

$$\frac{1}{2\pi}\left|\int_0^{2\pi} \gamma_k(c_k(\tilde{I}),\varphi)d\varphi - \int_0^{2\pi} \gamma_0(c_k(\tilde{I}),\varphi)d\varphi\right| = \left|\tilde{I} - \frac{1}{2\pi}\int_0^{2\pi} \gamma_0(c_k(\tilde{I}),\varphi)d\varphi\right| \leq \varepsilon.$$

From the above inequality we immediately obtain

$$\tilde{I} \leq \varepsilon + \frac{1}{2\pi}\int_0^{2\pi} \gamma_0(c_k(\tilde{I}), \varphi) d\varphi. \tag{49}$$

We proceed by considering the function

$$\tilde{I}(c) = \frac{1}{2\pi}\int_0^{2\pi} \gamma_0(c, \varphi) d\varphi,$$

which is strictly monotone for $c \geq \min f$. Moreover, if we fix $\varepsilon = \tilde{I}/2$, the inequality (49) gives

$$\frac{\tilde{I}}{2} \leq \tilde{I}(c_k(\tilde{I})) \tag{50}$$

$\forall k \geq K(\tilde{I}/2, c_*(\tilde{I}))$. Inequality (50) also implies that

$$c_k(\tilde{I}) > c(\tilde{I}/4)$$

$\forall k \geq K(\tilde{I}/2, c_*(\tilde{I}))$. In fact, if there exists $k \geq K(\tilde{I}/2, c_*(\tilde{I}))$ such that $c_k(\tilde{I}) \leq c(\tilde{I}/4)$, from (50) and the monotonicity of $\tilde{I}(c)$, we have also

$$\frac{\tilde{I}}{2} \leq \tilde{I}(c_k(\tilde{I})) \leq \tilde{I}\Big(c\Big(\frac{\tilde{I}}{4}\Big)\Big) = \frac{\tilde{I}}{4},$$

which is a contradiction. Moreover, since $\tilde{I} > 0$, one necessarily has $c(\tilde{I}/4) > \min f$.

(iii) We prove that, for any $c'' \geq c' > \min f$, we have

$$\Big|\tilde{I}(c'') - \tilde{I}(c')\Big| \geq \left|c'' - c'\right| \frac{m(c')}{2\sqrt{2}(c'' - \min f)}, \tag{51}$$

where $m(c')$ is the measure of the set

$$A_+(c') := \{\varphi \in \mathbb{S}^1 : c' \geq f(\varphi)\}.$$

Indeed, since $\tilde{I}(c)$ is a strictly monotone,

$$\begin{aligned}
\Big|\tilde{I}(c'') - \tilde{I}(c')\Big| &= \frac{1}{2\pi}\int_0^{2\pi} (\gamma_0(c'', \varphi) - \gamma_0(c', \varphi)) d\varphi \geq \\
&\geq \frac{1}{2\pi}\int_{A_+(c')} (\sqrt{2(c'' - f(\varphi))} - \sqrt{2(c' - f(\varphi))}) d\varphi = \\
&= \frac{|c'' - c'|}{\pi}\int_{A_+(c')} \frac{1}{\sqrt{2(c'' - f(\varphi))} + \sqrt{2(c' - f(\varphi))}} d\varphi \geq \\
&\geq \left|c'' - c'\right| \frac{m(c')}{2\sqrt{2}\pi(c'' - \min f)}.
\end{aligned} \tag{52}$$

(*iv*) Finally, from point (*ii*) we know that for any $\eta > 0$, there exists $K(\eta, c_*(\tilde{I}))$ such that, for any $k \geq K(\eta, c_*(\tilde{I}))$,

$$\left|\tilde{I}(c(\tilde{I})) - \tilde{I}(c_k(\tilde{I}))\right| = \left|\tilde{I} - \frac{1}{2\pi}\int_0^{2\pi} \gamma_0(c_k(\tilde{I}), \varphi) d\varphi\right| \leq \eta.$$

Moreover, since $c(\tilde{I}) > \min f$ and for any $k \geq K_2(\tilde{I})$ one also has $c_k(\tilde{I}) > c(I/4) > \min f$, we can apply inequality (51) to $c(\tilde{I})$ and $c_k(\tilde{I})$, obtaining

$$\left|\tilde{I}(c(\tilde{I})) - \tilde{I}(c_k(\tilde{I}))\right| \geq \left|c(\tilde{I}) - c_k(\tilde{I})\right| \frac{m(\min\{c(\tilde{I}), c_k(\tilde{I})\})}{2\sqrt{2}\pi(\max\{c(\tilde{I}), c_k(\tilde{I})\} - \min f)}$$

and therefore

$$\left|c(\tilde{I}) - c_k(\tilde{I})\right| \leq 2\sqrt{2}\pi\ \eta\ \frac{(\max\{c(\tilde{I}), c_k(\tilde{I})\} - \min f)}{m(\min\{c(\tilde{I}), c_k(\tilde{I})\})} \leq 2\sqrt{2}\pi\ \eta\ \frac{\max\{c(\tilde{I}), c_*\}}{m(c(\tilde{I}/4))}.$$

Therefore, setting

$$\eta = \varepsilon \frac{m(c(\tilde{I}/4))}{2\sqrt{2}\pi \max\{c(\tilde{I}), c_*(\tilde{I})\}}$$

for any $\varepsilon > 0$, we have proved that there exists $K_3(\varepsilon, \tilde{I})$ such that

$$\left|c(\tilde{I}) - c_k(\tilde{I})\right| \leq \varepsilon.$$

$\forall k \geq K_3(\varepsilon, \tilde{I})$.

5.4 Uniform convergence of u_k to u_0

In this section we prove that for any $\tilde{I} > 0$ and $\varepsilon > 0$ there exists $\tilde{K}(\varepsilon, \tilde{I})$ such that, for any $k \geq \tilde{K}(\varepsilon, \tilde{I})$, we have

$$|u_k(\tilde{I}, \varphi) - u_0(\tilde{I}, \varphi)| \leq \varepsilon$$

$\forall \varphi \in \mathbb{S}^1$.

The result follows from the estimates (I) and (II) to be established below.

(I) We know from (*i*) of Section 5.3 that there exists $c_*(\tilde{I})$ such that $\sup_k c_k(\tilde{I}) < c_*(\tilde{I})$. We can therefore apply the convergence result of Section 5.2 to conclude that for any $\eta > 0$ there exists $K(\eta, c_*(\tilde{I}))$ such that, for any $k \geq K(\eta, c_*(\tilde{I}))$, one has

$$|\gamma_k(c_k(\tilde{I}), \varphi) - \gamma_0(c_k(\tilde{I}), \varphi)| \leq \eta \tag{53}$$

$\forall \varphi \in \mathbb{S}^1$.

(II) For any $c \in \mathbb{R}$,

$$\lim_{c' \to c} \max_{\varphi \in \mathbb{S}^1} \left|\gamma_0(c', \varphi) - \gamma_0(c, \varphi)\right| = 0.$$

In other words, for any $\eta > 0$ there exists $\rho(\eta, c)$ such that for any c' with $|c' - c| \le \rho(\eta)$ and $\varphi \in \mathbb{S}^1$,

$$\left|\gamma_0(c', \varphi) - \gamma_0(c, \varphi)\right| \le \eta. \tag{54}$$

This is trivial if $c \le \min f$ or $c \ge \max f$. Let us therefore consider $c \in (\min f, \max f)$ and distinguish two cases.
(i) Let φ be such that $c - f(\varphi) \le \eta^2/32$, so that $\gamma_0(c, \varphi) \le \eta/4$. If $|c' - c| \le \eta^2/32$, then we have also $c' - f(\varphi) \le \eta^2/16$ and

$$\left|\gamma_0(c', \varphi) - \gamma_0(c, \varphi)\right| \le \gamma_0(c', \varphi) + \gamma_0(c, \varphi) \le \eta.$$

(ii) Let φ be such that $c - f(\varphi) \ge \eta^2/32$, so that $\gamma_0(c, \varphi) = \sqrt{2(c - f(\varphi))} \ge \eta/4$. If $|c' - c| \le \eta^2/64$, then we also have $c' - f(\varphi) \ge \eta^2/64$ and $\gamma_0(c', \varphi) = \sqrt{2(c' - f(\varphi))} \ge \eta/(4\sqrt{2})$. As a consequence

$$\left|\gamma_0(c', \varphi) - \gamma_0(c, \varphi)\right| \le \frac{2\,|c' - c|}{\sqrt{2(c - f(\varphi))} + \sqrt{2(c' - f(\varphi))}} \le \frac{2\frac{\eta^2}{64}}{\frac{\eta}{4} + \frac{\eta}{8}} \le \eta.$$

Therefore the uniform continuity is proved also for $c \in (\min f, \max f)$ with $\rho(\eta) = \eta^2/64$.

Hence

$$\begin{aligned}
|u_k(\tilde I, \varphi) - u_0(\tilde I, \varphi)| &= \left|\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\varphi}[\gamma_k(c_k(\tilde I), x) - \gamma_0(c(\tilde I), x)]dxd\varphi + \int_0^{\varphi}[\gamma_k(c_k(\tilde I), x) - \gamma_0(c(\tilde I), x)]dx\right| \\
&\le 4\pi \sup_{x \in \mathbb{S}^1}\left|\gamma_k(c_k(\tilde I), x) - \gamma_0(c(\tilde I), x)\right| \\
&\le 4\pi \sup_{x \in \mathbb{S}^1}\Big(|\gamma_k(c_k(\tilde I), x) - \gamma_0(c_k(\tilde I), x)| + |\gamma_0(c_k(\tilde I), x) - \gamma_0(c(\tilde I), x)|\Big).
\end{aligned}$$

From (I), for any $k \ge K(\varepsilon/8\pi, c_*(\tilde I))$, we have

$$4\pi \sup_{x \in \mathbb{S}^1}|\gamma_k(c_k(\tilde I), x) - \gamma_0(c_k(\tilde I), x)| \le \frac{\varepsilon}{2}.$$

Since $c_k(\tilde I)$ converges to $c(\tilde I)$, for any $k \ge K_3(\rho(\varepsilon/8\pi), \tilde I)$ we have $\left|c_k(\tilde I) - c(\tilde I)\right| \le \rho(\varepsilon/8\pi)$, and therefore by (II)

$$4\pi \sup_{x \in \mathbb{S}^1}|\gamma_0(c_k(\tilde I), x) - \gamma_0(c(\tilde I), x)| \le \frac{\varepsilon}{2}.$$

For $k \ge \tilde K(\varepsilon, \tilde I) = \max\{K(\varepsilon/8\pi, c_*(\tilde I)), K_3(\rho(\varepsilon/8\pi), \tilde I)\}$, we can therefore write

$$|u_k(\tilde I, \varphi) - u_0(\tilde I, \varphi)| \le \varepsilon$$

$\forall \varphi \in \mathbb{S}^1$, which establishes the statement on uniform convergence. $\square$

6 Proof of Theorem 4.1

Before treating the one dimensional case, we show that for general mechanical Hamiltonian systems with n degrees of freedom,

$$H(I,\varphi) = \sum_{i=1}^{n} \frac{I_i^2}{2} + f(\varphi_1,\ldots,\varphi_n), \tag{55}$$

the integrals in (18) and (19) can be respectively written in the form

$$E_1(k) = \int_{|\tilde{I}|\le R}\int_{\mathbb{T}^n} \Big|\frac{\partial}{\partial I}\Big[\frac{1}{2}(\tilde{I}+\frac{\partial u_k}{\partial\varphi})^2 - \bar{H}_k(\tilde{I})\Big]\Big|^2 \sigma_k d\varphi d\tilde{I} \tag{56}$$

and

$$E_2(k) = \int_{\mathbb{T}^n} \Big|\frac{\partial}{\partial\varphi}\Big[\frac{1}{2}(\tilde{I}+\frac{\partial u_k}{\partial\varphi})^2 + f(\varphi)\Big]\Big|^2 \sigma_k d\varphi. \tag{57}$$

Indeed, from (12), and with the notation of Section 2, we have

$$\dot{\tilde{\varphi}}^t = \dot{\varphi}^t + \frac{\partial^2 u_k}{\partial\varphi\partial I}(\tilde{I},\varphi^t)\dot{\varphi}^t.$$

Therefore, since for the Hamiltonian (55) the torus flow is

$$\dot{\varphi}^t = \tilde{I} + \frac{\partial u_k}{\partial\varphi}(\tilde{I},\varphi^t),$$

we have

$$\dot{\tilde{\varphi}}^t = (\mathbb{I}+\frac{\partial^2 u_k}{\partial\varphi\partial I})(\tilde{I}+\frac{\partial u_k}{\partial\varphi}) = \frac{\partial}{\partial I}\Big[\frac{1}{2}(\tilde{I}+\frac{\partial u_k}{\partial\varphi})^2\Big]_{|\varphi=\mathcal{C}^t_{\tilde{I}}(\varphi)}$$

from which we immediately obtain

$$\int_{|\tilde{I}|\le R}\int_{\mathbb{T}^n}\Big|\frac{\partial\tilde{\varphi}^t}{\partial t} - \frac{\partial\bar{H}_k}{\partial I}(\tilde{I})\Big|^2\sigma_k d\varphi d\tilde{I} = \int_{|\tilde{I}|\le R}\int_{\mathbb{T}^n}\Big|\frac{\partial}{\partial I}\Big[\frac{1}{2}(\tilde{I}+\frac{\partial u_k}{\partial\varphi})^2 - \bar{H}_k(\tilde{I})\Big]_{|\varphi=\mathcal{C}^t_{\tilde{I}}(\varphi)}\Big|^2\sigma_k d\varphi d\tilde{I}.$$

Since the torus flow preserves the measure defined by σ_k, see (14), we obtain (56).

To prove (57), on recalling (11):

$$I^t = \tilde{I} + \frac{\partial u_k}{\partial\varphi}(\tilde{I},\varphi^t),$$

we now obtain

$$\dot{I}^t = \frac{\partial^2 u_k}{\partial\varphi^2}(\tilde{I},\varphi^t)\dot{\varphi}^t = \frac{\partial^2 u_k}{\partial\varphi^2}(\tilde{I},\varphi^t)(\tilde{I}+\frac{\partial u_k}{\partial\varphi}(\tilde{I},\varphi^t)).$$

Moreover, for mechanical Hamiltonians (55),

$$\frac{\partial H}{\partial\varphi}(\Phi^t(\tilde{I},\varphi)) = \frac{\partial f}{\partial\varphi}(\varphi^t),$$

hence

$$\dot{I}^t + \frac{\partial H}{\partial\varphi}(\Phi^t(\tilde{I},\varphi)) = \frac{\partial^2 u_k}{\partial\varphi^2}(\tilde{I},\varphi^t)(\tilde{I}+\frac{\partial u_k}{\partial\varphi}(\tilde{I},\varphi^t)) + \frac{\partial f}{\partial\varphi}(\varphi^t) = \frac{\partial}{\partial\varphi}\Big[\frac{1}{2}(\tilde{I}+\frac{\partial u_k}{\partial\varphi})^2 + f(\varphi)\Big]_{|\varphi=\mathcal{C}^t_{\tilde{I}}(\varphi)}$$

and, again using (14), formula (57) is proved.

6.1 The one dimensional case

In the one dimensional case, for $\tilde{I} > 0$ one has $\gamma_k = \tilde{I} + \frac{\partial u_k}{\partial \varphi}$. Thus on invoking the symmetry with respect to $\tilde{I}$, formulas (56) and (57) become respectively

$$E_1(k) = \int_{-R}^{R}\int_0^{2\pi} \Big|\frac{\partial \tilde{\varphi}^t}{\partial t} - \frac{\partial \bar{H}_k}{\partial I}(\tilde{I})\Big|^2 \sigma_k d\varphi d\tilde{I} = 2\int_0^{R}\int_0^{2\pi} \Big|\frac{\partial}{\partial I}\Big[\frac{\gamma_k^2}{2} - \bar{H}_k(\tilde{I})\Big]\Big|^2 \sigma_k d\varphi d\tilde{I} \tag{58}$$

and

$$E_2(k) = \int_0^{2\pi} \Big|\frac{\partial I^t}{\partial t} + \frac{\partial H}{\partial \varphi}(\Phi^t(\tilde{I},\varphi))\Big|^2 d\varphi = \int_0^{2\pi} \Big|\frac{\partial}{\partial \varphi}\Big[\frac{\gamma_k^2}{2} + f(\varphi)\Big]\Big|^2 \sigma_k d\varphi. \tag{59}$$

6.2 Proof of (36) for $E_1(k)$

From (35) and (30) we obtain

$$\begin{aligned} E_1(k) &= 2\int_0^{R}\int_0^{2\pi} \Big|\frac{\partial}{\partial I}\Big[c_k(\tilde{I}) - \frac{1}{k}\log \gamma_k(c_k(\tilde{I}),\varphi) - \bar{H}_k(\tilde{I})\Big]\Big|^2 \sigma_k d\varphi d\tilde{I} = \\ &= 2\int_0^{R}\int_0^{2\pi} \Big|\frac{\partial}{\partial I}\Big[-\frac{1}{k}\log \gamma_k(c_k(\tilde{I}),\varphi) - \frac{1}{k}\log \int_0^{2\pi} \frac{dx}{\gamma_k(c_k(\tilde{I}),x)}\Big]\Big|^2 \sigma_k d\varphi d\tilde{I}, \end{aligned} \tag{60}$$

that is

$$E_1(k) = \frac{2}{k^2}\int_0^{R}\int_0^{2\pi} \Big|\frac{\gamma_k'(c_k,\varphi)}{\gamma_k(c_k,\varphi)} - \frac{\int_0^{2\pi} \frac{\gamma_k'(c_k,x)}{\gamma_k^2(c_k,x)}dx}{\int_0^{2\pi} \frac{dx}{\gamma_k(c_k,x)}}\Big|^2 \sigma_k d\varphi d\tilde{I},$$

where we have denoted

$$\gamma_k'(c_k,\varphi) := \frac{\partial}{\partial \tilde{I}}\gamma_k(c_k(\tilde{I}),\varphi), \qquad \gamma_k'(c_k,x) := \frac{\partial}{\partial \tilde{I}}\gamma_k(c_k(\tilde{I}),x).$$

Now, by expanding the square, using (28) and performing the integration over φ, we have

$$E_1(k) = \frac{2}{k^2}\int_0^{R} \frac{\int_0^{2\pi} \frac{{\gamma_k'}^2(c_k,x)}{\gamma_k^3(c_k,x)}dx \int_0^{2\pi} \frac{dx}{\gamma_k(c_k,x)} - \Big(\int_0^{2\pi} \frac{\gamma_k'(c_k,x)}{\gamma_k^2(c_k,x)}dx\Big)^2}{\Big(\int_0^{2\pi} \frac{dx}{\gamma_k(c_k,x)}\Big)^2} d\tilde{I}.$$

From (35) it is also easy to see that

$$\frac{d\gamma_k}{dc}\Big|_{c=c_k} = \frac{k\gamma_k}{k\gamma_k^2+1},$$

whence

$$\gamma_k'(c_k,x) = \frac{d\gamma_k}{dc}\Big|_{c=c_k} c_k' = \frac{\gamma_k(c_k,x)}{\frac{1}{k} + \gamma_k^2(c_k,x)} c_k'. \tag{61}$$

Using (61) we have the representation

$$E_1(k) = \frac{2}{k^2}\int_0^{R} {c_k'}^2(\tilde{I}) \frac{A_{-\frac{1}{2},2}(k,c_k(\tilde{I}))A_{-\frac{1}{2},0}(k,c_k(\tilde{I})) - A^2_{-\frac{1}{2},1}(k,c_k(\tilde{I}))}{A^2_{-\frac{1}{2},0}(k,c_k(\tilde{I}))} d\tilde{I}, \tag{62}$$

where the functions

$$A_{\alpha,\beta}(k,c) := \int_0^{2\pi} \frac{\gamma_k^{2\alpha}(c,\varphi)}{\left(\frac{1}{k}+\gamma_k^2(c,\varphi)\right)^{\beta}} d\varphi$$

are defined for any $c \in \mathbb{R}$.
Since $R > \tilde{I}(\max f + r) > 0$, we will proceed with the integral

$$\tilde{E}_1(k) = \frac{2}{k^2} \int_{\tilde{I}(\max f + r)}^{R} {c'_k}^2(\tilde{I}) \frac{A_{-\frac{1}{2},2}(k, c_k(\tilde{I}))A_{-\frac{1}{2},0}(k, c_k(\tilde{I})) - A^2_{-\frac{1}{2},1}(k, c_k(\tilde{I}))}{A^2_{-\frac{1}{2},0}(k, c_k(\tilde{I}))} d\tilde{I}, \tag{63}$$

since obviously $E_1(k) \geq \tilde{E}_1(k)$. From (33) it is also easy to obtain

$$c'_k(\tilde{I}) = \frac{2\pi}{\int_0^{2\pi} \frac{k\gamma_k}{k\gamma_k^2+1}} = \frac{2\pi}{A_{\frac{1}{2},1}(k, c_k(\tilde{I}))}. \tag{64}$$

Since $c'_k(\tilde{I}) > 0$ for any $\tilde{I} > 0$, in particular for $R > \tilde{I}(\max f + r) > 0$, we can change the integration variable in $\tilde{E}_1(k)$ from $\tilde{I}$ to $c = c_k(\tilde{I})$ and then invoking (64) we have

$$\tilde{E}_1(k) = \frac{4\pi}{k^2} \int_{c_k(\tilde{I}(\max f + r))}^{c_k(R)} \frac{A_{-\frac{1}{2},2}(k,c)A_{-\frac{1}{2},0}(k,c) - A^2_{-\frac{1}{2},1}(k,c)}{A^2_{-\frac{1}{2},0}(k,c)A_{\frac{1}{2},1}(k,c)} dc. \tag{65}$$

In order to estimate the last integral by the Dominated Convergence Theorem, we organize the proof into points (i) - (viii).

(i) Since

$$\lim_{k\to+\infty} c_k(R) = c(R), \qquad \lim_{k\to+\infty} c_k(\tilde{I}(\max f + r)) = c(\tilde{I}(\max f + r)) = \max f + r$$

there exist $K_4(r), K_5(R)$ such that, for any $k \geq K_4(r)$,

$$c_k(\tilde{I}(\max f + r)) > \max f + \frac{r}{2} \tag{66}$$

and for any $k \geq K_5(R)$,

$$c_k(R) < 2c(R). \tag{67}$$

(ii) For any $\varepsilon > 0$, $R > \tilde{I}(\max f + r)$, there exists $K_6(\varepsilon, R)$ such that, for any $k \geq K_6(\varepsilon, R)$ and $c \in [c_k(\tilde{I}(\max f + r)), c_k(R)]$,

$$|\gamma_k(c,\varphi) - \gamma_0(c,\varphi)| < \varepsilon \tag{68}$$

$\forall\varphi$. This follows from (67) and the uniform convergence of γ_k to γ_0 on the set $c \leq 2c(R)$, as soon as $K_6(\varepsilon, R) = \max\{K_5(R), K(\varepsilon, 2c(R))\}$.

(iii) For any $k \geq \max\{K_4(r), K_6(\sqrt{r}/2, R)\}$, for any $c \in [c_k(\tilde{I}(\max f + r)), c_k(R)]$ and any $\beta \geq 0$, we have

$$A_{-\frac{1}{2},\beta}(k,c) \leq 2\pi \Big(\frac{4}{r}\Big)^{\frac{1+2\beta}{2}}. \tag{69}$$

Indeed, since $k \geq K_6(\sqrt{r}/2, R)$, one has

$$|\gamma_k(c,\varphi) - \gamma_0(c,\varphi)| < \frac{\sqrt{r}}{2}$$

so that

$$\gamma_k(c,\varphi) \geq \gamma_0(c,\varphi) - \frac{\sqrt{r}}{2}$$

$\forall \varphi$. Since $c \geq c_k(\tilde{I}(\max f + r))$, if $k \geq K_4(r)$, we have also $c \geq c_k(\tilde{I}(\max f + r)) \geq \max f + r/2$, and therefore

$$\gamma_0(c,\varphi) = \sqrt{2(c - \max f)} \geq \sqrt{r},$$

hence

$$\gamma_k(c,\varphi) \geq \frac{\sqrt{r}}{2}.$$

As a consequence,

$$A_{-\frac{1}{2},\beta}(k,c) \leq \int_0^{2\pi} \frac{1}{\gamma_k(c,\varphi)^{1+2\beta}} d\varphi \leq 2\pi \Big(\frac{4}{r}\Big)^{\frac{1+2\beta}{2}}.$$

(iv) For any $k \geq \max\{K_4(r), K_5(R), K_6(\sqrt{r}/2, R)\}$ and $c \in [c_k(\tilde{I}(\max f + r)), c_k(R)]$, we have

$$A_{-\frac{1}{2},0}(k,c) \geq \frac{2\pi}{2\sqrt{c(R)} + \frac{\sqrt{r}}{2}}. \tag{70}$$

In fact, since $k \geq K_6(\sqrt{r}/2, R)$,

$$\gamma_k(c,\varphi) \leq \gamma_0(c,\varphi) + \frac{\sqrt{r}}{2},$$

and since $k \geq K_4(r)$ and $k \geq K_5(R)$,

$$\gamma_0(c,\varphi) = \sqrt{2(c - f(\varphi))} \leq \sqrt{2(2c(R))}.$$

Therefore

$$\gamma_k(c,\varphi) \leq 2\sqrt{c(R)} + \frac{\sqrt{r}}{2}.$$

(v) For any $k \geq \max\{K_4(r), K_5(R), K_6(\sqrt{r}/2, R)\}$ and $c \in [c_k(\tilde{I}(\max f + r)), c_k(R)]$, we have

$$A_{\frac{1}{2},1}(k,c) \geq \pi \frac{\sqrt{r}}{1 + (2\sqrt{c(R)} + \frac{\sqrt{r}}{2})^2}. \tag{71}$$

The proof is similar to points (iii) and (iv).

(vi) From (iii), (iv) and (v) it immediately follows that for any $k \geq \max\{K_4(r), K_5(R), K_6(\sqrt{r}/2, R)\}$ the integrand in (65):

$$\frac{A_{-\frac{1}{2},2}(k,c)A_{-\frac{1}{2},0}(k,c) - A^2_{-\frac{1}{2},1}(k,c)}{A^2_{-\frac{1}{2},0}(k,c)A_{\frac{1}{2},1}(k,c)}$$

is dominated by a constant independent of k, c.

(vii) To compute the pointwise limit of the integrand we consider the following:

Lemma 6.1 *Let $c > \max f$, $\lim_{k\to+\infty} c_k = c$ and $\beta \geq \alpha$. Then*

$$\lim_{k\to+\infty} A_{\alpha,\beta}(k,c) = a_{\beta-\alpha}(c), \tag{72}$$

where

$$a_\delta(c) := \int_0^{2\pi} \frac{1}{\gamma_0^{2\delta}(c,\varphi)} d\varphi = \int_0^{2\pi} \frac{1}{[2(c - f(\varphi))]^\delta} d\varphi.$$

Proof. This is an application of the Dominated Convergence Theorem. Indeed

$$\lim_{k\to+\infty} \frac{\gamma_k^{2\alpha}(c,\varphi)}{\left(\frac{1}{k} + \gamma_k^2(c,\varphi)\right)^\beta} = \frac{1}{\gamma_0^{2(\beta-\alpha)}(c,\varphi)}.$$

Moreover, the integrand in $A_{\alpha,\beta}(k,c)$:

$$\frac{\gamma_k^{2\alpha}(c,\varphi)}{\left(\frac{1}{k} + \gamma_k^2(c,\varphi)\right)^\beta} \leq \frac{1}{\gamma_k^{2(\beta-\alpha)}(c,\varphi)}$$

is dominated by a constant independent on k, φ for any $k \geq \max\{K_4(r), K_6(\sqrt{r}/2, R)\}$, see (iii). The statement now follows immediately.

(viii) Finally, in order to apply the Dominated Convergence Theorem to the integral in (65), we first write it in the following form

$$\int_{c_k(\tilde{I}(\max f+r))}^{c_k(R)} \frac{A_{-\frac{1}{2},2}(k,c)A_{-\frac{1}{2},0}(k,c) - A^2_{-\frac{1}{2},1}(k,c)}{A^2_{-\frac{1}{2},0}(k,c)A_{\frac{1}{2},1}(k,c)} dc =$$
$$= \int_{\mathbb{R}} \frac{A_{-\frac{1}{2},2}(k,c)A_{-\frac{1}{2},0}(k,c) - A^2_{-\frac{1}{2},1}(k,c)}{A^2_{-\frac{1}{2},0}(k,c)A_{\frac{1}{2},1}(k,c)} \chi_{\left[c_k(\tilde{I}(\max f+r)), c_k(R)\right]}(c) dc$$

and then we compute its pointwise limit (see Lemma 6.1).

$$\lim_{k\to+\infty} \frac{A_{-\frac{1}{2},2}(k,c)A_{-\frac{1}{2},0}(k,c) - A^2_{-\frac{1}{2},1}(k,c)}{A^2_{-\frac{1}{2},0}(k,c)A_{\frac{1}{2},1}(k,c)} \chi_{\left[c_k(\tilde{I}(\max f+r)), c_k(R)\right]}(c)$$

$$= \frac{-a^2_{\frac{3}{2}}(c) + a_{\frac{5}{2}}(c) a_{\frac{1}{2}}(c)}{a^3_{\frac{1}{2}}(c)} \chi_{[\max f+r, c(R)]}(c).$$

Consequently, defining the constant c_R as

$$c_R := 4\pi \int_{\max f + r}^{c(R)} \frac{-a^2_{\frac{3}{2}}(c) + a_{\frac{5}{2}}(c) a_{\frac{1}{2}}(c)}{a^3_{\frac{1}{2}}(c)} dc,$$

the estimate (36) follows. We observe that $c_R \neq 0$ for any non constant function f. Indeed, by using the L^2-Hölder inequality we obtain

$$\begin{aligned} a_{\frac{5}{2}}(c) a_{\frac{1}{2}(c)} &= \int_0^{2\pi} \left[\frac{1}{[2(c - f(\varphi))]^{5/4}}\right]^2 d\varphi \int_0^{2\pi} \left[\frac{1}{[2(c - f(\varphi))]^{1/4}}\right]^2 d\varphi \\ &\geq \left[\int_0^{2\pi} \frac{1}{[2(c - f(\varphi))]^{3/2}} d\varphi\right]^2 = a^2_{\frac{3}{2}}(c), \end{aligned}$$

hence $c_R \geq 0$, where the equality holds only if $f'(\varphi) = 0$ for any φ. □

6.3 Proof of (37) for $E_2(k)$

We know from (59) that

$$\begin{aligned} E_2(k) &= \int_0^{2\pi} \left| \frac{\partial}{\partial \varphi} \left[\frac{\gamma_k^2}{2}(c_k(\tilde{I}), \varphi) + f(\varphi)\right] \right|^2 \sigma_k(\tilde{I}, \varphi) d\varphi \\ &= \int_0^{2\pi} \left| \left[\gamma_k(c_k(\tilde{I}), \varphi) \frac{\partial}{\partial \varphi} \gamma_k(c_k(\tilde{I}), \varphi) + f'(\varphi)\right] \right|^2 \sigma_k(\tilde{I}, \varphi) d\varphi. \end{aligned}$$

By using the explicit expression for $\gamma_k(c_k(\tilde{I}), \varphi) = \sqrt{\frac{W(e^{2(c_k(\tilde{I}) - f(\varphi))k} k}{k}}$ and the derivative of the Lambert function

$$W'(z) = \frac{1}{(1 + W(z)) e^{W(z)}},$$

we first establish that

$$E_2(k) = \int_0^{2\pi} \left| \frac{f'(\varphi)}{1 + k\gamma_k^2(c_k(\tilde{I}), \varphi)} \right|^2 \sigma_k(c_k(\tilde{I}), \varphi) d\varphi.$$

Moreover, since

$$\sigma_k(c_k(\tilde{I}), \varphi) = \frac{1}{A_{-\frac{1}{2},0}(k, c_k(\tilde{I}))} \frac{1}{\gamma_k(c_k(\tilde{I}), \varphi)},$$

we finally obtain

$$E_2(k) = \frac{1}{k^2} \frac{1}{A_{-\frac{1}{2},0}(k, c_k(\tilde{I}))} \int_0^{2\pi} \frac{|f'(\varphi)|^2}{\gamma_k(c_k(\tilde{I}), \varphi) \left(\frac{1}{k} + \gamma_k^2(c_k(\tilde{I}), \varphi)\right)^2} d\varphi. \tag{73}$$

In order to estimate $\lim_{k \to +\infty} k^2 E_2(k)$ by the Dominated Convergence Theorem, we proceed as follows.

(i) Since $\sup_k c_k(\tilde{I}) < c_*(\tilde{I})$, for any $\varepsilon > 0$ there exists $K(\frac{\varepsilon}{2}, c_*(\tilde{I}))$ such that

$$|\gamma_k(c_k(\tilde{I}),\varphi) - \gamma_0(c_k(\tilde{I}),\varphi)| \le \frac{\varepsilon}{2}$$

$\forall k > K(\frac{\varepsilon}{2}, c_*(\tilde{I}))$ and $\varphi \in \mathbb{S}^1$. In particular

$$\gamma_k(c_k(\tilde{I}),\varphi) \ge \gamma_0(c_k(\tilde{I}),\varphi) - \frac{\varepsilon}{2}$$

$\forall k > K(\frac{\varepsilon}{2}, c_*(\tilde{I}))$ and $\varphi \in \mathbb{S}^1$.

(ii) Moreover, we know that for any $\eta > 0$ there exists $\rho(\eta) > 0$ such that, for any c' with $|c' - c| \le \rho(\eta)$, we have

$$|\gamma_0(c',\varphi) - \gamma_0(c,\varphi)| \le \eta$$

$\forall \varphi \in \mathbb{S}^1$. Therefore, since $\lim_{k\to+\infty} c_k(\tilde{I}) = c(\tilde{I})$, we have

$$|c_k(\tilde{I}) - c(\tilde{I})| \le \rho(\frac{\varepsilon}{2})$$

$\forall k > K_3(\rho(\frac{\varepsilon}{2}), \tilde{I})$. Hence

$$|\gamma_0(c_k(\tilde{I}),\varphi) - \gamma_0(c(\tilde{I}),\varphi)| \le \frac{\varepsilon}{2}$$

$\forall k > K_3(\rho(\frac{\varepsilon}{2}), \tilde{I})$ and $\varphi \in \mathbb{S}^1$. As a consequence,

$$\gamma_0(c_k(\tilde{I}),\varphi) \ge \gamma_0(c(\tilde{I}),\varphi) - \frac{\varepsilon}{2}$$

$\forall k > K_3(\rho(\frac{\varepsilon}{2}), \tilde{I})$ and $\varphi \in \mathbb{S}^1$.

(iii) Finally, since $c(\tilde{I}) > \max f$ and $\lim_{k\to+\infty} c_k(\tilde{I}) = c(\tilde{I})$, for any $\tilde{I} > 0$ there exists $k(\tilde{I})$ such that

$$c_k(\tilde{I}) > \max f$$

$\forall k > k(\tilde{I})$.

By using (i) and (ii), we conclude that for any $k > \max\{K(\frac{\varepsilon}{2}, c_*(\tilde{I})), K_3(\rho(\frac{\varepsilon}{2}), \tilde{I})\}$ and $\varphi \in \mathbb{S}^1$, we have

$$|\gamma_k(c_k(\tilde{I}),\varphi) - \gamma_0(c(\tilde{I}),\varphi| \le |\gamma_k(c_k(\tilde{I}),\varphi) - \gamma_0(c_k(\tilde{I}),\varphi| + |\gamma_0(c_k(\tilde{I}),\varphi) - \gamma_0(c(\tilde{I}),\varphi| \le \varepsilon.$$

In particular,

$$\lim_{k\to+\infty} \gamma_k(c_k(\tilde{I}),\varphi) = \gamma_0(c(\tilde{I}),\varphi) \tag{74}$$

$\forall \varphi \in \mathbb{S}^1$. Moreover, for any $k > \max\{K(\frac{\varepsilon}{2}, c_*(\tilde{I})), K_3(\rho(\frac{\varepsilon}{2}), \tilde{I}), k(\tilde{I})\}$ and $\varphi \in \mathbb{S}^1$,

$$\gamma_k(c_k(\tilde{I}),\varphi) \ge \gamma_0(c_k(\tilde{I}),\varphi) - \frac{\varepsilon}{2} \ge \gamma_0(c(\tilde{I}),\varphi) - \varepsilon \ge \sqrt{2(c(\tilde{I}) - \max f)} - \varepsilon. \tag{75}$$

In the last inequalities, since $k > k(\tilde{I})$, the function $\gamma_0(c_k(\tilde{I}), \varphi) = \sqrt{2(c_k(\tilde{I}) - f(\varphi))} > 0$ for any $\varphi \in \mathbb{S}^1$ and therefore the constant $\sqrt{2(c(\tilde{I}) - \max f)} - \varepsilon$ (independent of φ and k) is positive (choose, for example, $\varepsilon = \frac{\sqrt{2(c(\tilde{I}) - \max f)}}{2}$).
Finally we apply the Dominated Convergence Theorem to compute $\lim_{k\to+\infty} k^2 E_2(k)$, where $E_2(k)$ is given by the expression in (73).
From (75), we immediately obtain

$$\frac{|f'(\varphi)|^2}{\gamma_k(c_k(\tilde{I}, \varphi))\left(\frac{1}{k} + \gamma_k^2(c_k(\tilde{I}, \varphi))\right)^2} \le \frac{\max_{\varphi\in\mathbb{S}^1} |f'(\varphi)|^2}{\gamma_k^5(c_k(\tilde{I}), \varphi))} \le \frac{\max_{\varphi\in\mathbb{S}^1} |f'(\varphi)|^2}{\left(\sqrt{2(c(\tilde{I}) - \max f)} - \varepsilon\right)^5}.$$

Moreover, from (74),

$$\lim_{k\to+\infty} \frac{|f'(\varphi)|^2}{\gamma_k(c_k(\tilde{I}), \varphi)\left(\frac{1}{k} + \gamma_k^2(c_k(\tilde{I}), \varphi)\right)^2} = \frac{|f'(\varphi)|^2}{\gamma_0^5(c(\tilde{I}), \varphi)}.$$

Therefore

$$\lim_{k\to+\infty} \int_0^{2\pi} \frac{|f'(\varphi)|^2}{\gamma_k(c_k(\tilde{I}), \varphi)\left(\frac{1}{k} + \gamma_k^2(c_k(\tilde{I}), \varphi)\right)^2} d\varphi = \int_0^{2\pi} \frac{|f'(\varphi)|^2}{\gamma_0^5(c(\tilde{I}), \varphi)} d\varphi. \tag{76}$$

We will conclude by proving that

$$\lim_{k\to+\infty} A_{-\frac{1}{2},0}(k, c_k(\tilde{I})) = a_{\frac{1}{2}}(c(\tilde{I})), \tag{77}$$

where

$$A_{-\frac{1}{2},0}(k, c_k(\tilde{I})) = \int_0^{2\pi} \frac{1}{\gamma_k(c_k(\tilde{I}), \varphi)} d\varphi \quad \text{and} \quad a_{\frac{1}{2}}(c(\tilde{I})) = \int_0^{2\pi} \frac{1}{[2(c(\tilde{I}) - f(\varphi))]^{1/2}} d\varphi.$$

This limit is again a straightforward consequence of the Dominated Convergence Theorem. In fact (see (75) and (74)),

$$\frac{1}{\gamma_k(c_k(\tilde{I}), \varphi)} \le \frac{1}{\sqrt{2(c(\tilde{I}) - \max f)} - \varepsilon},$$

where the right hand member is independent of φ and k, and

$$\lim_{k\to+\infty} \frac{1}{\gamma_k(c_k(\tilde{I}), \varphi)} = \frac{1}{\gamma_0(c(\tilde{I}), \varphi)}.$$

From (76) and (77) we therefore obtain the result

$$\lim_{k\to+\infty} k^2 E_2(k) = \frac{1}{a_{\frac{1}{2}}(c(\tilde{I}))} \int_0^{2\pi} \frac{|f'(\varphi)|^2}{\gamma_0^5(c(\tilde{I}), \varphi)} d\varphi = \frac{1}{a_{\frac{1}{2}}(c(\tilde{I}))} \int_0^{2\pi} \frac{|f'(\varphi)|^2}{[2(c(\tilde{I}) - f(\varphi)]^{5/2}} d\varphi.$$

7 Proof of Proposition 3.3

We start by introducing some notation: $x_k(t)$ and $x(t)$ denote respectively the solutions of

$$\dot{x}_k = \gamma_k(c_k(\tilde{I}), x_k) \qquad \text{and} \quad \dot{x} = \gamma_0(c(\tilde{I}), x), \tag{78}$$

with $x_k(0) = x(0) = 0$. The function $x_k(t)$ is periodic, with period

$$T_k := \int_0^{2\pi} \frac{1}{\gamma_k(c_k(\tilde{I}), x)} dx.$$

Lemma 7.1 *If $c(\tilde{I}) = \max f$, we have*

$$\lim_{k\to\infty} T_k = +\infty.$$

Proof. We first consider the trivial estimate

$$\int_0^{2\pi} \frac{1}{\gamma_k(c_k(\tilde{I}), x)} dx \geq \int_{f(x)\leq c_k(\tilde{I})} \frac{1}{\gamma_k(c_k(\tilde{I}), x)} dx,$$

and then we use Lemma 5.1 to estimate the right hand side integral. In fact, for any x such that $f(x) \leq c_k(\tilde{I})$, it follows from Lemma 5.1 that there exist $0 < a_1 < 1 < a_2$ and $\tilde{K}_1(a_1, a_2)$ such that

$$\frac{a_1}{\tilde{\gamma}_0(c_k(\tilde{I}), x)} \leq \frac{1}{\gamma_k(c_k(\tilde{I}), x)} \leq \frac{a_2}{\tilde{\gamma}_0(c_k(\tilde{I}), x)}$$

for any $k > \tilde{K}_1(a_1, a_2)$, with $\tilde{\gamma}_0(c_k(\tilde{I}), x) := \sqrt{2(c_k(\tilde{I}) - f(x)) + \frac{\log k}{k}}$. Therefore, for such k we also have

$$T_k \geq a_1 \int_{f(x)\leq c_k(\tilde{I})} \frac{1}{\sqrt{2(c_k(\tilde{I}) - f(x)) + \frac{\log k}{k}}} dx.$$

Moreover, from Theorem 3.2, point (iii), for any $\varepsilon > 0$ there exists $\tilde{K}_2(\tilde{I}, \varepsilon)$ such that for any $k > \tilde{K}_2(\tilde{I}, \varepsilon)$ it holds $c_k(\tilde{I}) \leq c(\tilde{I}) + \varepsilon$. Therefore, for any $k > \max\{\tilde{K}_1(a_1, a_2), \tilde{K}_2(\tilde{I}, \varepsilon)\}$ we have also

$$T_k \geq a_1 \int_0^{2\pi} \frac{1}{\sqrt{2(c(\tilde{I}) + \varepsilon - f(x)) + \frac{\log k}{k}}} \chi_{f(x)\leq c_k(\tilde{I})}(x) dx,$$

where $\chi_{f(x)\leq c_k(\tilde{I})}(x)$ denotes the characteristic function of the set $f(x) \leq c_k(\tilde{I})$.
Since $2(c(\tilde{I}) + \varepsilon - f(x)) + \frac{\log k}{k} \geq \varepsilon$, the integrand is dominated by a constant on $[0, 2\pi]$. Therefore, by the Dominated Convergence Theorem, we obtain

$$\lim_{k\to+\infty} \int_0^{2\pi} \frac{1}{\sqrt{2(c(\tilde{I}) + \varepsilon - f(x)) + \frac{\log k}{k}}} \chi_{f(x)\leq c_k(\tilde{I})} dx = \int_0^{2\pi} \frac{1}{\sqrt{2(c(\tilde{I}) + \varepsilon - f(x))}} dx = T(c(\tilde{I}) + \varepsilon),$$

where $T(c(\tilde{I}) + \varepsilon)$ is the period of

$$\dot{x} = \gamma_0(c(\tilde{I}) + \varepsilon, x).$$

Hence, for any $\varepsilon > 0$, there exists $\tilde{K}_3(\tilde{I}, \varepsilon)$ such that, for any $k > \tilde{K}_3(\tilde{I}, \varepsilon)$ we have

$$\int_0^{2\pi} \frac{1}{\sqrt{2(c(\tilde{I}) + \varepsilon - f(x)) + \frac{\log k}{k}}} \chi_{f(x) \le c_k(\tilde{I})} dx \ge \frac{1}{2} T(c(\tilde{I}) + \varepsilon),$$

and therefore, for any $\varepsilon > 0$ if $k > \max\{\tilde{K}_1(a_1, a_2), \tilde{K}_2(\tilde{I}, \varepsilon), \tilde{K}_3(\tilde{I}, \varepsilon)\}$,

$$T_k \ge \frac{a_1}{2}\, T(c(\tilde{I}) + \varepsilon).$$

Since $c(\tilde{I}) = \max f$, one has

$$\lim_{\varepsilon \to 0^+} T(c(\tilde{I}) + \varepsilon) = +\infty.$$

Therefore, for any $\eta > 0$ there exists $\varepsilon(\eta)$ such that

$$\frac{a_1}{2}\, T(c(\tilde{I}) + \varepsilon(\eta)) > \eta,$$

and also for any $k > \max\{\tilde{K}_1(a_1, a_2), \tilde{K}_2(\tilde{I}, \varepsilon(\eta)), \tilde{K}_3(\tilde{I}, \varepsilon(\eta))\}$, we have

$$T_k \ge \frac{a_1}{2}\, T(c(\tilde{I}) + \varepsilon(\eta)) > \eta.$$

Hence we have proved

$$\lim_{k \to +\infty} T_k = +\infty.$$

□

The rest of the proof will be formulated for $f(\varphi) = -\cos(\varphi)$.

Lemma 7.2 *Let $f(\varphi) = -\cos\varphi$ and $\tilde{I} \in (0, +\infty)$ be such that $c(\tilde{I}) = \max f = 1$. Then, for any $t \in \mathbb{R}$, we have*

$$\lim_{k \to +\infty} x_k(t) = x(t).$$

Proof. Let us denote $d_k(t) = |x_k(t) - x(t)|$. In order to overcome the lack of differentiability of d_k, for a constant $r > 0$, we introduce $e_k(t) = \sqrt{(x_k(t) - x(t))^2 + r^2}$, whose time derivative $\dot{e}_k(t)$ satisfies

$$\begin{aligned} \dot{e}_k(t) \;&\le\; \left|\gamma_k(c_k(\tilde{I}), x_k) - \gamma_0(c(\tilde{I}), x)\right| \le \\ &\le\; \left|\gamma_k(c_k(\tilde{I}), x_k) - \gamma_0(c_k(\tilde{I}), x_k)\right| + \left|\gamma_0(c_k(\tilde{I}), x_k) - \gamma_0(c(\tilde{I}), x_k)\right| + \left|\gamma_0(c(\tilde{I}), x_k) - \gamma_0(c(\tilde{I}), x)\right|. \end{aligned}$$

In the sequel we denote $c_k := c_k(\tilde{I})$, $c := c(\tilde{I})$ and we fix $t > 0$. By the uniform convergence of γ_k to γ_0, for any $\varepsilon > 0$ there exists $\tilde{K}_1(\varepsilon)$ such that for any $k > \tilde{K}_1(\varepsilon)$, we have

$$|\gamma_k(c_k, x_k(\tau)) - \gamma_0(c_k, x_k(\tau))| \le \varepsilon$$

for any $\tau \in \mathbb{R}$, and specifically for any $\tau \in [0, t]$.
From Lemma 7.1 and the convergence of c_k to c, there also exist $\tilde{K}_2(t)$, $\rho(t) > 0$ such that for $k > \tilde{K}_2(t)$

one has: $T_k/2 > t$, $|x_k(t) - \pi| > \rho(t)$ and also $f(x_k(\tau)) < c_k$ for any $\tau \le t$. Therefore, there exists $\lambda(t) < \infty$ such that

$$\sup_k \sup_{\tau \le t} \frac{1}{\gamma_0(c, x_k(\tau))} = \lambda(t).$$

As a consequence, we have

$$|\gamma_0(c_k, x_k(\tau)) - \gamma_0(c, x_k(\tau))| = \frac{|2(c_k - c)|}{\gamma_0(c_k, x_k(\tau)) + \gamma_0(c, x_k(\tau))} \le \frac{|2(c_k - c)|}{\gamma_0(c, x_k(\tau))} \le 2\lambda(t)\,|c_k - c|\,.$$

By the convergence of c_k to c, for for any $\varepsilon > 0$ there exists $\tilde{K}_3(\varepsilon)$ such that, for any $k > \tilde{K}_3(\varepsilon)$, $|2(c_k - c)| \le \varepsilon$, and therefore for any $k > \max\{\tilde{K}_2(t), \tilde{K}_3(\varepsilon)\}$ and any $\tau \in [0, t]$,

$$|\gamma_0(c_k, x_k(\tau)) - \gamma_0(c, x_k(\tau))| \le \lambda(t)\varepsilon.$$

Moreover, since 1 is a Lipschitz constant for f, we have

$$|\gamma_0(c, x_k(\tau)) - \gamma_0(c, x(\tau))| \le \frac{|2(f(x_k(\tau)) - f(x(\tau)))|}{\gamma_0(c, x_k(\tau)) + \gamma_0(c, x(\tau))} \le \frac{|2(x_k(\tau) - x(\tau))|}{\gamma_0(c, x_k(\tau))} \le 2\lambda(t) d_k(\tau).$$

Therefore, for any $\tau \le t$, for any $\varepsilon > 0$ and any $k > \max\{\tilde{K}_1(\varepsilon), \tilde{K}_2(t), \tilde{K}_3(\varepsilon)\}$, one has

$$\dot{e}_k(\tau) \le (1 + \lambda(t))\varepsilon + 2\lambda(t) d_k(\tau).$$

Since $d_k < e_k$, by the Gronwall Lemma, we have

$$e_k(\tau) \le \frac{(1 + \lambda(t))\varepsilon}{2\lambda(t)} \left(e^{2\lambda(t)\tau} - 1\right),$$

and also

$$d_k(\tau) \le \frac{(1 + \lambda(t))\varepsilon}{2\lambda(t)} \left(e^{2\lambda(t)t} - 1\right).$$

For any $\eta > 0$, let us consider $\varepsilon(\eta)$ such that $\frac{(1+\lambda(t))\varepsilon}{2\lambda(t)} \left(e^{2\lambda(t)t} - 1\right) = \eta$. Then, for any $\eta > 0$ and any $k > \max\{\tilde{K}_1(\varepsilon(\eta)), \tilde{K}_2(t), \tilde{K}_3(\varepsilon(\eta))\}$, one has

$$d_k(\tau) \le e_k(\tau) \le \eta,$$

that is

$$\lim_{k \to \infty} d_k(\tau) = 0.$$

Therefore, the function $d_k(t)$ converges pointwise to 0 for any $t \in \mathbb{R}$, and the lemma is proved. □

Lemma 7.3 *Let $f(x) = -\cos x$, and $\tilde{I}$ such that $c(\tilde{I}) = 1$. Then, for any $\tilde{t} \in (0, 1/2)$, we have*

$$\lim_{k \to +\infty} x_k(\tilde{t}\, T_k) = \pi.$$

Proof. Since

$$\lim_{t\to+\infty} x(t) = \pi$$

for any $\varepsilon > 0$ there exists $T(\varepsilon)$ such for $t \geq T(\varepsilon)$ one has

$$|x(t) - \pi| \leq \varepsilon.$$

Let us now consider $t = T(\varepsilon)$. By Lemma 7.2 we have

$$\lim_{k\to+\infty} x_k(t) = x(t).$$

For any $\varepsilon > 0$ there exists $\tilde{K}_1(\varepsilon)$ such that, for any $k > \tilde{K}_1(\varepsilon)$, we have

$$|x_k(T(\varepsilon)) - \pi| \leq 2\varepsilon.$$

Let us now fix $\tilde{t} \in (0, 1/2)$, and consider $\tilde{K}_2(\tilde{t}, \varepsilon)$ such that for any $k > K_2(\tilde{t}, \varepsilon)$, $\tilde{t}T_k > t$ (this is possible since by Lemma 7.1 T_k is a divergent sequence). Since $x_k(\tau)$ is monotone, from the inequalities $t < \tilde{t}T_k < T_k/2$, we obtain

$$x_k(t) \leq x_k(\tilde{t}T_k) \leq x_k(T_k/2) = \pi,$$

and therefore, for any $k > \max\{\tilde{K}_1(\varepsilon), K_2(\tilde{t}, \varepsilon)\}$,

$$\left|x_k(\tilde{t}T_k) - \pi\right| \leq 2\varepsilon.$$

The lemma is therefore proved. □

We can now prove the Proposition 3.3. We consider the integral

$$\int_{-\pi}^{\pi} u(x)\sigma_k(x)dx.$$

and we change the integration variable from x to t by using $x = x_k(t)$, and then from t to $\tilde{t} = t/T_k$, thus obtaining

$$\int_{-\pi}^{\pi} u(x)\sigma_k(x)dx = \frac{1}{T_k}\int_{-\frac{T_k}{2}}^{\frac{T_k}{2}} u(x_k(t))dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} u(x_k(\tilde{t}\; T_k))d\tilde{t}.$$

We observe that, for any $\tilde{t} \in (0, 1/2)$,

$$\lim_{k\to+\infty} u(x_k(\tilde{t}\; T_k)) = u(\pi).$$

Also, for any $t \in (-1/2, 0)$,

$$\lim_{k\to\infty} u(x_k(\tilde{t}\; T_k)) = u(-\pi) = u(\pi).$$

Therefore, the Lemma follows by the Dominated Convergence Theorem. □

8 The Lambert function W

The Lambert function W is defined as the multivalued function defined implicitly by the relation:

$$z = W(z)e^{W(z)} \tag{79}$$

for any complex number z. We only consider W for $z \in [0, +\infty)$, so that it becomes single valued. In particular, $W(z) \geq 0$ for $z \in [0, +\infty)$ and W is an increasing function.

The asymptotic properties of W may be characterized by asymptotic developments. We refer to [3] and [4] for all details and proofs. From these developments, one immediately obtains

$$\lim_{z \to +\infty} \frac{W(z)}{\log z - \log \log z} = 1 \tag{80}$$

and also

$$\lim_{z \to 0^+} \frac{W(z)}{z - z^2} = 1. \tag{81}$$

Formulas (79), (80) and (81) are used extensively throughout this paper to define and prove the asymptotic properties of the key functions introduced in Definition 3.1.

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